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Minimum embedding of a P_4-design into a balanced incomplete block design of index λ

Peter Danziger [a], Salvatore Milici [b,*], Gaetano Quattrocchi [b]

[a] Department of Mathematics, Ryerson University, Toronto, ON M5B 2K3, Canada

[b] Dipartimento di Matematica e Informatica, Università di Catania, Catania, Italy

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ABSTRACT

Let H be a subgraph of G. An H-design $(V, \mathcal{C})$ of order v and index μ is *embedded* into a G-design $(X, \mathcal{B})$ of order $v + w$ and index λ if $\mu \le \lambda$, $V \subseteq X$ and there is an injective mapping $f : \mathcal{C} \to \mathcal{B}$ such that B is subgraph of $f(B)$ for every $B \in \mathcal{C}$.

For every pair of positive integers v, λ, (except when $\lambda = 3$ and $v = 30, 34, 42, 46, 54, 58, 66$ or $\lambda = 5$ and $v = 19$) we determine the minimum value of w such that there exists a balanced incomplete block design of order $v + w$, index λ and block-size 4 which embeds a P_4-design of order v and index $\mu = 1$ (P_4 denotes the path of length 3).

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1. Introduction and definitions

Let G be a finite and simple graph. A G-design of order v and index λ is a pair $(V, \mathcal{C})$ where V is the vertex set of K_v (the complete graph on v vertices) and $\mathcal{C}$ is a collection of isomorphic copies of the graph G, called *blocks*, which partition the edges of λK_v (the complete multigraph on v vertices).

A K_4-design of order v and index λ is well-known as a balanced incomplete block design of order v, index λ and block-size 4. We denote such a design as $S_\lambda(2, 4, v)$. Hanani [8] proved that an $S_\lambda(2, 4, v)$ exists if and only if:

- $v \equiv 1, 4 \pmod{12}$ if $\lambda \equiv 1, 5 \pmod 6$;
- $v \equiv 1 \pmod 3$ if $\lambda \equiv 2, 4 \pmod 6$;
- $v \equiv 0, 1 \pmod 4$ if $\lambda \equiv 3 \pmod 6$;
- any $v \ge 4$ if $\lambda \equiv 0 \pmod 6$.

A *path design* $P(v, s, \lambda)$ is a P_s-design of index λ, where P_s is the simple path with $s - 1$ edges, $[a_1, a_2, \ldots, a_s] = \{\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_{s-1}, a_s\}\}$. Tarsi [17] proved that a $P(v, s, \lambda)$ exists if and only if $s - 1 \mid \lambda\binom{v}{2}$.

Definition 1.1. Let H be a subgraph of G, and let $V \subseteq X$. We say that an H-design $(V, \mathcal{C})$ of order v and index μ is embedded into a G-design $(X, \mathcal{B})$ of order $v + w$ and index λ, $\mu \le \lambda$, if there is an injective mapping

$$f : \mathcal{C} \to \mathcal{B}$$

such that B is a subgraph of $f(B)$ for every $B \in \mathcal{C}$.

The mapping f is called the embedding of $(V, \mathcal{C})$ into $(X, \mathcal{B})$. When w attains the minimum possible value we say that f is a minimum embedding.

If $H = G$ and $\mu = \lambda$ then we obtain the usual embedding definition for G-designs.

* Corresponding author.
E-mail addresses: danziger@ryerson.ca (P. Danziger), milici@dmi.unict.it (S. Milici), quattrocchi@dmi.unict.it (G. Quattrocchi).

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Table 1
Necessary conditions for embedding, $\lambda \geq 3$

v (mod 12)	$v \geq$	λ (mod 6) ≥ 3	w
0, 1, 4, 9	4	3	0
3, 7	7	3	1
6, 10	18 for $\lambda = 3$, 6 for $\lambda \geq 9$	3	2
0, 3, 6, 9	6	2, 4	1
1, 4, 7, 10	4	2, 4	0
0, 3	12	1, 5	1
1, 4	4	1, 5	0
6	6	1, 5	7
7	7	1, 5	6
9	9	1, 5	4
10	10	1, 5	3
No constraint	4	0	0

When $\mu = \lambda = 1$, the (minimum) embedding of an H-design into a G-design has been studied for many pairs of graphs H and G with H a subgraph of G [2–4,6,9–12,14–16]. The well-known definition of complementary path decompositions of the complete multigraph can be restated as the minimum embedding of a $P(v, 4, \lambda)$ into an $S_{2\lambda}(2, 4, v)$. This problem has been completely solved by Granville, Moisiadis and Rees [7] for $\lambda = 1$ and Du [5] for $\lambda \geq 2$.

Recently Milici [13] proved the existence of a minimum embedding of a $P(v, 3, 1)$ into an $S_\lambda(2, 3, v + w)$ for every $v \equiv 0, 1 \pmod 4$ and every $\lambda \geq 2$. In this paper we wish to consider the next step, i.e the minimum embedding of a $P(v, 4, 1)$ into an $S_\lambda(2, 4, v + w)$. In particular, we will prove the following results.

Theorem 1.1 (*Main Theorem*).

- *There exists a minimum embedding of a $P(v, 4, 1)$ into an $S_2(2, 4, v + w)$ if and only if $v \equiv 1 \pmod 3$ and $w = 0$.*
- *If there exists a minimum embedding of a $P(v, 4, 1)$ into an $S_\lambda(2, 4, v + w)$ with $\lambda \geq 3$ then w satisfies the conditions given in Table 1 for every admissible v, except that when $\lambda = 3$, $w = 10$ for $v = 6$ and $w = 6$ for $v = 10$.*
  The above necessary conditions are sufficient except possibly when $\lambda = 3$ and $v = 30, 34, 42, 46, 54, 58, 66$ or $\lambda = 5$ and $v = 19$.

For $\lambda = 2$ consider an embedding f of a $P(v, 4, 1)$, $(V, \mathcal{C})$, into an $S_2(2, 4, v+w)$, $(V \cup W, \mathcal{B})$, with $|V \cap W| = 0$, $|W| = w$ if $w \geq 1$ and $W = \emptyset$ if $w = 0$. Then $(V, \mathcal{D})$, with $\mathcal{D} = \{[c, a, d, b] \mid [a, b, c, d] \in \mathcal{C}\}$, is another $P(v, 4, 1)$ embedded into $(V \cup W, \mathcal{B})$. The decomposition of $2K_v$ induced by $\mathcal{C} \cup \mathcal{D}$ is well-known as a complementary path decomposition. It is known that such a decomposition exists if and only if $v \equiv 1 \pmod 3$ [7]. An embedding of a $P(v, 4, 1)$ into an $S_2(2, 4, v)$ may be obtained by reversing the decomposition on one of the complementary path designs. Moreover, $(V, \{f(B) \mid B \in \mathcal{C}\})$ is an $S_2(2, 4, v)$ embedded into $(V \cup W, \mathcal{B})$. This completes the proof when $\lambda = 2$.

For $\lambda \geq 3$, other than the case $\lambda = 3$, $v \neq 6, 10$, the necessary part of the Main Theorem 1.1 follows easily from the necessary and sufficient conditions for the existence of a $P(v, 4, 1)$ and an $S_\lambda(2, 4, v + w)$. When $\lambda = 3$ and $v = 6, 10$, let $(V, \mathcal{C})$ be a $P(6, 4, 1)$ embedded into an $S_3(2, 4, 6 + w)(V \cup W, \mathcal{B})$. Counting the number of blocks of $\mathcal{B}$ meeting the vertices of W and the number of edges in V not covered by blocks inducing a path of $\mathcal{C}$, we get $w \geq 10$. Analogously we obtain $w \geq 6$ for $v = 10$ and $\lambda = 3$.

In the next section we consider some useful definitions and results and discuss the constructions used. In Section 3 we prove the sufficiency part of the Main Theorem 1.1 when $\lambda \geq 3$ by showing each of the cases $\lambda = 3, 4, 5, 6$, each case is dealt with in a separate subsection. These proofs draw heavily from the individual results presented in the Appendix. Many of the embeddings given in the Appendix are nontrivial and are found by a variety of methods.

2. Constructions and related structures

In this section we briefly discuss some of the constructions that we will use and introduce some useful definitions and results. For terms not defined in this paper or results not explicitly cited the reader is referred to the *CRC Handbook of Combinatorial Designs* [1] and its online updates.

A *pairwise balanced design* PBD(v, K) of order v with block-sizes from K is a pair $(V, \mathcal{B})$, where V is a finite set of cardinality v and $\mathcal{B}$ is a family of subsets of V (*blocks*) such that $|B| \in K$ for every $B \in \mathcal{B}$ and every pair of distinct elements of V occurs in exactly one block of $\mathcal{B}$. Pairwise balanced design constructions are widely used.

Theorem 2.1. *If there exists a* PBD(v, K) *and for each $k \in K$ there is an $S_\lambda(2, 4, k)$ which embeds a $P(k, 4, 1)$ then there is a an $S_\lambda(2, 4, v)$ which embeds a $P(v, 4, 1)$.*

Proof. The design is obtained in by placing an $S_\lambda(2, 4, k)$ which embeds a $P(k, 4, 1)$ on each block of size k. ■

A 4-GDD$_\lambda$ is a triple $(V, \mathcal{G}, B)$, where V is a finite set, $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ is a partition of V into subsets, the elements of $\mathcal{G}$ are called *groups*, and $\mathcal{B}$ is a collection of isomorphic copies of K_4, called *blocks*, which partition the edges of $K_{g_1, g_2, \ldots, g_n}$ ($|G_i| = g_i$), on the vertex set V. If for $i = 1, 2, \ldots, t$, there are u_i groups of size g_i, we say that the 4-GDD is of type $g_1^{u_1} g_2^{u_2} \ldots g_t^{u_t}$.

Let $\mathcal{B}$ be the block set of a design or a GDD. A *parallel class* or *resolution class* is a collection of blocks which partition the point set of the design or the GDD. A design or a GDD is *resolvable* if $\mathcal{B}$ can be partitioned into parallel classes.

We recall the existence of some 4-GDD and 4-RGDD we need in the following.

Lemma 2.2 ([1]). *There exists a* 4-GDD *of type:*

- 12^t6^1 *for each* $t \geq 2$;
- $12^t, 9^16^t, 18^112^t$ *for each* $t \geq 4$;
- $6^t, 24^112^t$ *for each* $t \geq 5$;
- 18^16^t *for each* $t \geq 7$;
- 4^11^t *for each* $t \equiv 0, 9 \pmod{12}$, $t \geq 9$;
- 7^11^t *for each* $t \equiv 0, 3 \pmod{12}$, $t \geq 15$;
- 10^11^t *for each* $t \equiv 0, 9 \pmod{12}$, $t \geq 21$;
- 13^11^t *for each* $t \geq 27$;
- 16^11^t *for each* $t \equiv 0, 9 \pmod{12}$, $t \geq 33$;
- 19^11^t *for each* $t \equiv 0, 3 \pmod{12}$, $t \geq 39$;
- 22^11^t *for each* $t \equiv 0, 9 \pmod{12}$, $t \geq 45$;
- 25^11^t *for each* $t \equiv 0, 3 \pmod{12}$, $t \geq 51$;
- 28^11^t *for each* $t \equiv 0, 9 \pmod{12}$, $t \geq 57$;
- 37^11^t *for each* $t \equiv 0 \pmod{12}$, $t \geq 75$;
- 49^11^t *for each* $t \equiv 0, 3 \pmod{12}$, $t \geq 99$.

There exists a resolvable 3-GDD *of type* 6^t *for each* $t \geq 4$ *and of type* 12^t *for each* $t \geq 3$.

We may interpret the groups of a GDD as blocks which may generate PBDs for construction 2.1. However, using the GDD's structure allows for more powerful constructions. We may add a point, ∞, to the point set of the GDD, the blocks are treated as in the PBD construction above, but on the groups the embedding is placed in such a way that the extra point is always ∞.

Theorem 2.3. *If there exist a k-GDD of type $g_i^{u_i}$ and an $S_\lambda(2, 4, k)$ which embeds a $P(k, 4, 1)$, and for each g_i we have an $S_\lambda(2, 4, g_i + 1)$ which embeds a $P(g_i, 4, 1)$ we have an $S_\lambda(2, 4, v + 1)$ which embeds a $P(v, 4, 1)$, where $v = \sum_i g_iu_i$.*

Ideally we would like to be able to add more than one point at a time, we can do this by considering embeddings with holes. Alternately, we allow an intermediate step in which certain points of a GDD appear a different number of times in the blocks.

Theorem 2.4. *If there exist a k-GDD of type h^1g^u; a k-GDD of type $(h+\lambda w)^1g^u$; an $S_\mu(2, 4, k)$ which embeds a $P(k, 4, 1)$, where $\mu < \lambda$; an $S_\lambda(2, 4, g)$ which embeds a $P(g, 4, 1)$ and an $S_\lambda(2, 4, h + w)$ which embeds a $P(h, 4, 1)$, then there exist a and an $S_\lambda(2, 4, v + w)$ which embeds a $P(v, 4, 1)$, where $v = h + gu$.*

Proof. Take a k-GDD of type h^1g^u, $(V, \{G, G_1, \dots, G_u\}, \mathcal{B})$ where G is the group of size h and $G_1, \dots, G_u$ are the groups of size g. Let $W = \{a_i \mid i \in \mathbb{Z}_w\}$, we will create the $S_\lambda(2, 4, v + w)$ on the point set $V \cup W$.

On each block $B \in \mathcal{B}$ place an $S_\mu(2, 4, k)$, $(B, \mathcal{B}_B)$, which embeds a $P(k, 4, 1)$. For each i place an $S_\lambda(2, 4, g)$, $(G_i, \mathcal{D}_i)$, which embeds a $P(g, 4, 1)$ with point set G_i. On $G \cup W$ place a $S_\lambda(2, 4, h + w)$, $(G \cup W, \mathcal{D})$ which embeds a $P(h, 4, 1)$. Repeating the blocks of the GDD $\lambda - \mu - 1$ times gives a set of blocks $\mathcal{F} = (\lambda - \mu - 1)\mathcal{B} \cup \mathcal{D} \cup \left(\bigcup_{i=1}^u \mathcal{D}_i\right) \cup \left(\bigcup_{B\in\mathcal{B}} \mathcal{B}_B\right)$ which embeds a $P(gu + h, 4, 1)$. Further in $\mathcal{F}$, every pair of points in the same group appears λ times together, every pair of points from different groups appears $\lambda - 1$ times. In addition, every point of $G \cup W$ appears λ times with the points of W but no other points appear with any points from W.

Let $(V \cup X, \{G \cup X, G_1, \dots G_u\}, \mathcal{C})$ be a k-GDD of type $(h+\lambda w)^1g^u$, where $X = \{\infty_{ij} \mid i \in \mathbb{Z}_w, j \in \mathbb{Z}_\lambda\}$. Consider the effect of the map $\sigma : X \to W$ given by $\sigma(\infty_{ij}) = a_i$ on the blocks $\mathcal{C}$. The blocks $\sigma(\mathcal{C})$ have each $a_i \in W$ appearing λ times with every point not in G. All other pairs of points are unaffected and appear as before pairs of points from the same group do not appear together, pairs from different groups appear exactly once. Thus the union $\mathcal{F} \cup \sigma(\mathcal{C})$ provides the required design. ∎

In particular, we note that when $k = 4$ we have an $S_\mu(2, 4, 4)$ which embeds a $P(4, 4, 1)$ for $\mu = 2, 4$. Also note that when $g = 1$ an $S_\lambda(2, 4, 1)$ which embeds a $P(1, 4, 1)$ is trivial. We will also use some variations on this construction which will be explained as needed.

3. Proof of the Main theorem

We now complete the proof of the Main Theorem 1.1 by solving the cases $\lambda = 3, 4, 5, 6$. We consider each case in the subsections below. It is easy to see that the sufficiency of the Main Theorem for $\lambda = 3, 4, 5, 6$ implies its sufficiency for every λ. For example, the minimum embedding of a $P(v, 4, 1)$ into an $S_{6k+1}(2, 4, v + w)$, with $k \geq 1$, is obtained by pasting the blocks of an $S_5(2, 4, v + w)$ which embeds a $P(v, 4, 1)$ to the blocks of an $S_2(2, 4, v + w)$ and if $k \geq 2$ also, by pasting the blocks of an $S_{6(k-1)}(2, 4, v + w)$.

3.1. $\lambda = 3$

Theorem 3.1. *If* $v \equiv 0, 1, 4, 9 \pmod{12}$ *then there is an* $S_3(2, 4, v)$ *which embeds a* $P(v, 4, 1)$.

Proof. For $v = 4, 9, 12, 21, 24$ see steps 1, 4, 6, 11 and 13 respectively in the Appendix. For $v \geq 13$, $v \neq 21, 24$, apply Theorem 2.1 using a PBD$(v, \{4, 9, 12\})$ which can be found in [1]. ■

Theorem 3.2. *If* $v \equiv 3 \pmod{12}$, $v \geq 15$ *then there is an* $S_3(2, 4, v + 1)$ *which embeds a* $P(v, 4, 1)$.

Proof. For $v = 15, 27, 39$ see steps 7, 14 and 15 in the Appendix.

For $v \geq 51$ we use a holey construction. Take a 4-GDD $(X, \mathcal{G}, \mathcal{B})$ of type 12^t, $t \geq 4$. On each $B \in \mathcal{B}$, place an $S_3(2, 4, 4)$ which embeds a $P(4, 4, 1)$. Let $V = X \cup Y \cup \{\infty\}$, where $Y = \{y_1, y_2, y_3\}$ and $|(Y \cup \{\infty\}) \cap X| = 0$. Fix $G \in \mathcal{G}$. On $G \cup Y \cup \{\infty\}$, place an $S_3(2, 4, 16)$ which embeds a $P(15, 4, 1)$ on the vertex set $G \cup Y$. On $D \cup Y \cup \{\infty\}$, $D \in \mathcal{G}$ $D \neq G$, place a K_4-decomposition of $3(K_{16} \setminus K_4)$ having $Y \cup \{\infty\}$ as the hole and embedding a P_4-decomposition of $K_{15} \setminus K_3$ on the vertex set $D \cup Y$ with Y as the hole (see step 8 in the Appendix). ■

Theorem 3.3. *If* $v \equiv 6 \pmod{12}$ *there exist an* $S_3(2, 4, 16)$ *and an* $S_9(2, 4, 8)$ *which embeds a* $P(6, 4, 1)$ *and for* $v \geq 18$, $v \neq 30, 42, 54, 66$, *there is an* $S_3(2, 4, v + 2)$ *which embeds a* $P(v, 4, 1)$.

Proof. Steps 2 and 46 in the Appendix give an $S_3(2, 4, 16)$ and an $S_9(2, 4, 8)$ which embed a $P(6, 4, 1)$, respectively. For $v = 18$ an $S_3(2, 4, 20)$ which embeds a $P(18, 4, 1)$ is given by step 9 in the Appendix.

For $v \geq 78$, use Theorem 2.4 with $w = \mu = 2$, $h = 18$ and $g = 12$. A 4-GDD of type $18^1 12^t$ and a 4-GDD of type $24^1 12^t$, $t \geq 5$ are given in Lemma 2.2. An $S_3(2, 4, 12)$ which embeds a $P(12, 4, 1)$ is given by Theorem 3.1. ■

Note that, in Theorem 3.3, the cases $v = 30, 42, 54, 66$ remain open.

Theorem 3.4. *If* $v \equiv 7 \pmod{12}$, $v \geq 15$ *then there is an* $S_3(2, 4, v + 1)$ *which embeds a* $P(v, 4, 1)$.

Proof. For $v = 7, 19$ see steps 3 and 10 in the Appendix. For $v \geq 31$ use Theorem 2.4 with $w = 1$, $\mu = 2$, $h = 7$ and $g = 1$. A 4-GDD of type $7^1 1^{12t}$ and a 4-GDD of type $10^1 1^{12t}$, $t \geq 2$ are given in Lemma 2.2. ■

Theorem 3.5. *If* $v \equiv 10 \pmod{12}$, $v \neq 34, 46, 58$ *then there exist an* $S_3(2, 4, 16)$ *and an* $S_9(2, 4, 12)$ *which embeds a* $P(10, 4, 1)$. *For* $v \geq 22$, *there is an* $S_3(2, 4, v + 2)$ *which embeds a* $P(v, 4, 1)$.

Proof. Steps 5 and 47 in the Appendix give an $S_3(2, 4, 16)$ and an $S_9(2, 4, 12)$ which embed a $P(10, 4, 1)$, respectively. For $v = 22, 70$ see steps 12 and 16 in the Appendix. For $v \geq 82$ use Theorem 2.4 with $w = 2$, $h = 22$ and $g = 1$. A 4-GDD of type $22^1 1^{12t}$ and a 4-GDD of type $28^1 1^{12t}$, $t \geq 2$ are given in Lemma 2.2. ■

Note that in Theorem 3.5 the cases $v = 34, 46, 58$ remain open.

3.2. $\lambda = 4$

For $v \equiv 1 \pmod 3$ the proof of the Main Theorem follows by doubling the solution for $\lambda = 2$. So we suppose $v \equiv 0 \pmod 3$.

Theorem 3.6. *If* $v \equiv 0 \pmod 6$, $v \geq 6$ *then there is an* $S_4(2, 4, v + 1)$ *which embeds a* $P(v, 4, 1)$.

Proof. For $v = 6, 12, 18, 24$ see steps 17, 21, 23 and 25 in the Appendix. For $v \geq 30$ apply Theorem 2.3, using a 4-GDD of type 6^t, $t \geq 5$ from Lemma 2.2. ■

Theorem 3.7. *If* $v \equiv 3 \pmod 6$, $v \geq 6$ *then there is an* $S_4(2, 4, v + 1)$ *which embeds a* $P(v, 4, 1)$.

Proof. For $v = 9, 15, 21, 27$ see steps 19, 22, 24 and 26 in the Appendix. For $v \geq 33$ apply Theorem 2.3 using a 4-GDD of type $9^1 6^t$, $t \geq 4$. ■

3.3. $\lambda = 5$

For $v \equiv 1, 4 \pmod{12}$ the proof of the Main Theorem follows by pasting an $S_3(2, 4, v)$ to an $S_2(2, 4, v)$ which embeds a $P(v, 4, 1)$. For $v \equiv 0, 3 \pmod{12}$ we get the proof by pasting an $S(2, 4, v + 1)$ to an $S_4(2, 4, v + 1)$ which embeds a $P(v, 4, 1)$.

Since w for the remaining cases is relatively large we cannot use Theorem 2.4 directly, but rather use some interesting variations on it in each case.

Theorem 3.8. *If* $v \equiv 6 \pmod{12}$, $v \geq 6$ *then there is an* $S_5(2, 4, v + 7)$ *which embeds a* $P(v, 4, 1)$.

Proof. For $v = 6, 18, 30, 42, 54, 66, 78$ see steps 27, 31, 35, 37, 39, 40, 41 in the Appendix. For $v \geq 90$, take a 4-GDD $(V, \mathcal{D})$ of type 6^{1+2t}, $t \geq 7$, having groups G_i, $i = 1, 2, \ldots, 1+2t$. On each block of $\mathcal{D}$ place an $S_2(2, 4, 4)$ which embeds a $P(4, 4, 1)$.

Construct an $S_5(2,4,13)$ on $G_1 \cup \{a_0, a_1, \ldots, a_6\}$ which embeds a $P(6,4,1)$ on G_1. For each $i = 2, 3, \ldots, 1+2t$, place an $S_4(2,4,7)$ on $G_i \cup \{a_6\}$ which embeds a $P(6,4,1)$ having G_i as the vertex set.

Now take another 4-GDD of type 6^{1+2t} having groups $\overline{G}_1 = \{a_0, a_1, \ldots, a_6\}$ and $\overline{G}_i = G_i$, $i = 2, 3, \ldots, 1+2t$ and repeat its blocks twice.

Finally, take a 4-GDD of type $37^1 1^{12t}$ having $F = \{\infty_{ij}, \overline{\infty}_{ij} \mid (i,j) \in \mathbb{Z}_6 \times \mathbb{Z}_3\} \cup \{a_6\}$ as a group of size 37 and such that the groups of size 1 cover $V \setminus G_1$. For every pair $(i,j) \in \mathbb{Z}_6 \times \mathbb{Z}_3$, replace ∞_{ij} with i and $\overline{\infty}_{ij}$ with a_i and take the blocks so obtained. ■

Theorem 3.9. *If $v \equiv 7 \pmod{12}$, $v \geq 7$, $v \neq 19$ then there is an $S_5(2,4,v+6)$ which embeds a $P(v,4,1)$.*

Proof. For $v = 7, 31, 43$ see steps 28, 36 and 38 in the Appendix. Now let $v = 1 + 12t \geq 55$, so there exists a 4-GDD $(V, \mathcal{D})$ of type $7^1 1^{12t}$, $t \geq 4$, on $V = \mathbb{Z}_v$ having $G = \mathbb{Z}_7$ as a group of size 7. On each block of $\mathcal{D}$ place an $S_2(2,4,4)$ which embeds a $P(4,4,1)$. Now take a 4-GDD of type $19^1 1^{12t}$, $t \geq 4$, on $\mathbb{Z}_v \cup \{\infty_{ij} \mid (i,j) \in \mathbb{Z}_6 \times \mathbb{Z}_2\}$ having $G \cup \{\infty_{ij} \mid (i,j) \in \mathbb{Z}_6 \times \mathbb{Z}_2\}$ as a group of size 19. For each $i \in \mathbb{Z}_6$, replace ∞_{ij} with a_i and repeat the blocks so obtained twice. On $G \cup \{a_0, a_1, \ldots, a_5\}$, place an $S_4(2,4,13)$ which embeds a $P(7,4,1)$ having G as the vertex set (see step 18 in the Appendix). The result is an $S_4(2,4,v+6)$ which embeds a $P(v,4,1)$. Paste an $S(2,4,v+6)$ on $V \cup \{a_0, a_1, \ldots, a_5\}$. ■

Note that in Theorem 3.9, the case $v = 19$ remains open.

Theorem 3.10. *If $v \equiv 9 \pmod{12}$, $v \geq 9$ then there is an $S_5(2,4,v+4)$ which embeds a $P(v,4,1)$.*

Proof. For $v = 9, 21$ see 29 and 32 in the Appendix. Let $v \geq 33$ and take an embedding of a $P(v,4,1)$ on $V = \mathbb{Z}_v$ into an $S_3(2,4,v)$. Now take a 4-GDD $(X, \mathcal{D})$ of type $16^1 1^v$, with point set $X = \mathbb{Z}_v \cup \{\infty_{ij} \mid (i,j) \in \mathbb{Z}_4 \times \mathbb{Z}_4\}$ and such that $H = \{\infty_{ij} \mid (i,j) \in \mathbb{Z}_4 \times \mathbb{Z}_4\}$ is the group of size 16. For each $(i,j) \in \mathbb{Z}_4 \times \mathbb{Z}_4$, replace ∞_{ij} with a_i and take the blocks so obtained. Finally, paste an $S_4(2,4,4)$ on $\{a_0, a_1, a_2, a_3\}$ and an $S(2,4,v+4)$ on $\mathbb{Z}_v \cup \{a_0, a_1, a_2, a_3\}$. ■

Theorem 3.11. *If $v \equiv 10 \pmod{12}$, $v \geq 10$ then there is an $S_5(2,4,v+3)$ which embeds a $P(v,4,1)$.*

Proof. For $v = 10, 22$ see steps 30 and 33 in the Appendix. Let $v = 10 + 12t \geq 34$ and take an embedding of a $P(v,4,1)$ into an $S_2(2,4,v)$ on $\mathbb{Z}_v$. Now take a 4-GDD of type $10^1 1^{12t+9}$, $t \geq 1$, on the vertex set $\mathbb{Z}_v \cup \{\infty_{ij} \mid (i,j) \in \mathbb{Z}_3 \times \mathbb{Z}_3\}$ having $G = \{\infty_{ij} \mid (i,j) \in \mathbb{Z}_3 \times \mathbb{Z}_3\} \cup \{v-1\}$ as a group of size 10. For each $(i,j) \in \mathbb{Z}_3 \times \mathbb{Z}_3$ replace ∞_{ij} with a_i and take the blocks so obtained. Place an $S_3(2,4,4)$ on $\{a_0, a_1, a_2, v-1\}$. The result is an $S_3(2,4,v+3)$ which embeds a $P(v,4,1)$. Paste an $S_2(2,4,v+3)$ on $\mathbb{Z}_v \cup \{a_0, a_1, a_2\}$. ■

3.4. $\lambda = 6$

For $v \equiv 1 \pmod 3$ ($v \equiv 0, 9 \pmod{12}$) the proof of the Main Theorem follows by tripling (doubling) the solution for $\lambda = 2$ ($\lambda = 3$). So we suppose $v \equiv 3, 6 \pmod{12}$.

Theorem 3.12. *If $v \equiv 3, 6 \pmod{12}$, $v \geq 6$ then there is an $S_6(2,4,v)$ which embeds a $P(v,4,1)$.*

Proof. For $v = 6, 15, 18, 27$ see steps 42, 43, 44 and 45 in the Appendix. For the rest of the values we may use Theorem 2.1. When $v \equiv 3 \pmod{12}$ take a 4-GDD of type $9^1 6^{2t-1}$, $t \geq 3$, and consider the groups as blocks to obtain a PBD$(12t+3, \{4,6,9\})$. When $v \equiv 6 \pmod{12}$ take a 4-GDD of type 6^{2t-1}, $t \geq 3$, and consider the groups as blocks to obtain a PBD$(12t-6, \{4,6\})$. ■

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Appendix

In this Appendix we list some minimum embeddings of a $P(v,4,1)(V, \mathcal{C})$ into an $S_\lambda(2,4,v)(V \cup W, \mathcal{B})$ for small values of v. We use the following notation: when V or W is not specified we suppose $V = \mathbb{Z}_v$ or $V = \mathbb{Z}_{v-1} \cup \{\infty\}$ and $W = \{a_0, a_1, \ldots, a_{w-1}\}$ if $w \geq 1$ or $W = \emptyset$ if $w = 0$. We list *only* the blocks of $\mathcal{B}$, using square brackets (braces) if the block is (is not) in $\mathcal{C}$. For example, $[x, y, z, t]$ means that the K_4 on vertices x, y, z, t is a block of $\mathcal{B}$ and that the P_4 having the same vertices and edges $\{x, y\}$, $\{y, z\}$ and $\{z, t\}$ is a block of $\mathcal{C}$; whereas $\{x, y, z, t\}$ denotes a block of $\mathcal{B}$ not inducing a path in $\mathcal{C}$. When we list base blocks for $\mathcal{B}$ we intend them to be developed $\pmod v$ ($\pmod{v-1}$) where the vertex set is $\mathbb{Z}_v$ ($\mathbb{Z}_{v-1} \cup \{\infty\}$).

1. $\lambda = 3$, $v = 4$, $w = 0$. Blocks: $[0, 1, 2, 3]$, $[1, 3, 0, 2]$, $\{0, 1, 2, 3\}$.

2. $\lambda = 3$, $v = 6$, $w = 10$. Develop (mod 5) the base block $[\infty, 0, 1, 3]$. Add the following blocks: $\{0, 1, a_5, a_0\}$; $\{1, 2, a_6, a_1\}$; $\{2, 3, a_7, a_2\}$; $\{3, 4, a_8, a_3\}$; $\{4, 0, a_9, a_4\}$; $\{0, 1, a_9, a_3\}$; $\{1, 2, a_5, a_4\}$; $\{2, 3, a_6, a_0\}$; $\{3, 4, a_7, a_1\}$; $\{4, 0, a_8, a_2\}$; $\{0, 2, a_8, a_1\}$; $\{1, 3, a_9, a_1\}$; $\{2, 4, a_5, a_3\}$; $\{3, 0, a_6, a_4\}$; $\{4, 1, a_7, a_5\}$; $\{0, a_0, a_2, a_7\}$; $\{0, a_0, a_3, a_8\}$; $\{0, a_1, a_4, a_9\}$; $\{0, a_1, a_7, a_5\}$; $\{0, a_2, a_5, a_6\}$; $\{0, a_3, a_7, a_6\}$; $\{1, a_2, a_4, a_8\}$; $\{1, a_0, a_1, a_5\}$; $\{1, a_3, a_4, a_7\}$; $\{1, a_1, a_6, a_8\}$; $\{1, a_3, a_8, a_7\}$; $\{1, a_2, a_6, a_9\}$; $\{2, a_0, a_4, a_8\}$; $\{2, a_1, a_2, a_7\}$; $\{2, a_2, a_3, a_6\}$; $\{2, a_0, a_9, a_8\}$; $\{2, a_3, a_9, a_5\}$; $\{2, a_4, a_9, a_7\}$; $\{3, a_2, a_3, a_5\}$; $\{3, a_1, a_4, a_5\}$; $\{3, a_0, a_3, a_6\}$; $\{3, a_0, a_9, a_5\}$; $\{3, a_1, a_9, a_8\}$; $\{3, a_4, a_7, a_8\}$; $\{4, a_0, a_2, a_9\}$; $\{4, a_1, a_3, a_9\}$; $\{4, a_4, a_1, a_6\}$; $\{4, a_0, a_7, a_6\}$; $\{4, a_2, a_5, a_8\}$; $\{4, a_4, a_5, a_6\}$; $\{\infty, a_0, a_1, a_2\}$; $\{\infty, a_1, a_2, a_3\}$; $\{\infty, a_2, a_3, a_4\}$; $\{\infty, a_3, a_4, a_0\}$; $\{\infty, a_4, a_0, a_1\}$; $\{\infty, a_5, a_6, a_8\}$; $\{\infty, a_6, a_7, a_9\}$; $\{\infty, a_7, a_8, a_5\}$; $\{\infty, a_8, a_9, a_6\}$; $\{\infty, a_9, a_5, a_7\}$.
3. $\lambda = 3$, $v = 7$, $w = 1$. Base blocks: $[0, 1, 3, 6]$, $\{a_0, 0, 1, 3\}$.
4. $\lambda = 3$, $v = 9$, $w = 0$. Blocks: $[3, 1, 2, 4]$, $[0, 1, 8, 3]$, $[5, 1, 4, 7]$, $[6, 1, 7, 0]$, $[7, 6, 2, 3]$, $[5, 2, 0, 6]$, $[2, 8, 4, 6]$, $[2, 7, 8, 5]$, $[0, 4, 3, 6]$, $[8, 6, 5, 3]$, $[0, 3, 7, 5]$, $[8, 0, 5, 4]$, $\{0, 1, 2, 8\}$, $\{1, 4, 5, 6\}$, $\{1, 2, 3, 5\}$, $\{1, 6, 7, 8\}$, $\{0, 2, 4, 7\}$, $\{3, 4, 7, 8\}$.
5. $\lambda = 3$, $v = 10$, $w = 6$. Blocks: the blocks of an $S_2(2, 4, 10)$ which embeds a $P(10, 4, 1)$ (see Section 3) and the following ones. $\{a_0, a_1, 7, 8\}$, $\{a_0, a_2, 8, 9\}$, $\{a_0, a_3, 9, 5\}$, $\{a_0, a_4, 5, 6\}$, $\{a_0, a_5, 6, 7\}$, $\{a_0, a_1, 2, 3\}$, $\{a_0, a_2, 3, 4\}$, $\{a_0, a_3, 4, 0\}$, $\{a_0, a_4, 0, 1\}$, $\{a_0, a_5, 1, 2\}$, $\{a_0, a_1, 3, 9\}$, $\{a_0, a_2, 4, 5\}$, $\{a_0, a_3, 0, 6\}$, $\{a_0, a_4, 1, 7\}$, $\{a_0, a_5, 2, 8\}$, $\{a_1, a_3, 7, 9\}$, $\{a_2, a_4, 8, 5\}$, $\{a_3, a_5, 9, 6\}$, $\{a_4, a_1, 5, 7\}$, $\{a_5, a_2, 6, 8\}$, $\{a_1, a_3, 0, 8\}$, $\{a_2, a_4, 1, 9\}$, $\{a_3, a_5, 2, 5\}$, $\{a_4, a_1, 3, 6\}$, $\{a_5, a_2, 4, 7\}$, $\{a_1, a_2, 0, 2\}$, $\{a_2, a_3, 1, 3\}$, $\{a_3, a_4, 2, 4\}$, $\{a_4, a_5, 3, 0\}$, $\{a_5, a_1, 4, 1\}$, $\{a_1, a_2, 2, 6\}$, $\{a_2, a_3, 3, 7\}$, $\{a_3, a_4, 4, 8\}$, $\{a_4, a_5, 0, 9\}$, $\{a_5, a_1, 1, 5\}$, $\{a_1, a_2, 0, 5\}$, $\{a_2, a_3, 1, 6\}$, $\{a_3, a_4, 2, 7\}$, $\{a_4, a_5, 3, 8\}$, $\{a_5, a_1, 4, 9\}$.
6. $\lambda = 3$, $v = 12$, $w = 0$. Base blocks: $[\infty, 1, 2, 4]$, $[1, 4, 8, 3]$, $\{0, 1, 4, 6\}$.
7. $\lambda = 3$, $v = 15$, $w = 1$. Blocks (all sums are (mod 15)): $[4 + i, 6 + i, 12 + i, i]$, $i \in \mathbb{Z}_{15}$; $[1 + 3i, 6 + 3i, 7 + 3i, 8 + 3i]$, $[3 + 3i, 2 + 3i, 6 + 3i, 13 + 3i]$, $i \in \mathbb{Z}_5$; $[12 + i, 1 + i, 8 + i, 13 + i]$, $\{2 + i, 7 + i, 8 + i, 9 + i\}$, $i \in \mathbb{Z}_{15} \setminus \{2, 5, 8, 11, 14\}$; $\{a_0, i, 2 + i, 5 + i\}$, $i \in \mathbb{Z}_{15}$.
8. A K_4-decomposition of $3(K_{16} \setminus K_4)$ (having vertex set $\mathbb{Z}_{15} \cup \{a_0\}$ and hole $\{a_0, 0, 1, 2\}$) which embeds a P_4-decomposition of $K_{15} \setminus K_3$ having vertex set $\mathbb{Z}_{15}$ and hole $\{0, 1, 2\}$. Blocks: $[0, 4, 14, 8]$, $[0, 6, 3, 10]$, $[0, 9, 7, 12]$, $[0, 5, 11, 13]$, $[0, 3, 9, 14]$, $[0, 10, 4, 13]$, $[0, 7, 5, 12]$, $[0, 8, 6, 11]$, $[0, 11, 3, 7]$, $[0, 12, 4, 8]$, $[0, 13, 5, 9]$, $[0, 14, 6, 10]$, $[1, 8, 3, 13]$, $[1, 14, 7, 4]$, $[1, 11, 10, 5]$, $[1, 12, 9, 6]$, $[4, 1, 7, 11]$, $[5, 1, 6, 12]$, $[9, 1, 3, 14]$, $[10, 1, 13, 8]$, $[2, 5, 4, 3]$, $[2, 10, 9, 11]$, $[2, 14, 13, 12]$, $[2, 9, 8, 5]$, $[8, 2, 7, 6]$, $[4, 2, 12, 10]$, $[6, 2, 11, 14]$, $[3, 2, 13, 7]$, $[3, 5, 6, 4]$, $[4, 9, 13, 6]$, $[4, 11, 12, 14]$, $[13, 10, 8, 7]$, $[3, 12, 8, 11]$, $[5, 14, 10, 7]$, $\{7, 8, 9, 10\}$, $\{11, 12, 13, 14\}$, $\{3, 5, 6, 9\}$, $\{1, 13, 12, 6\}$, $\{1, 10, 11, 5\}$, $\{1, 7, 14, 9\}$, $\{1, 8, 4, 3\}$, $\{2, 3, 10, 12\}$, $\{2, 4, 9, 11\}$, $\{2, 5, 8, 14\}$, $\{2, 6, 7, 13\}$, $\{a_0, 13, 14, 3\}$, $\{a_0, 10, 4, 9\}$, $\{a_0, 7, 12, 5\}$, $\{a_0, 8, 11, 6\}$, $\{a_0, 13, 4, 5\}$, $\{a_0, 10, 14, 6\}$, $\{a_0, 7, 11, 3\}$, $\{a_0, 8, 12, 9\}$, $\{a_0, 13, 11, 9\}$, $\{a_0, 10, 12, 3\}$, $\{a_0, 7, 4, 6\}$, $\{a_0, 8, 14, 5\}$.
9. $\lambda = 3$, $v = 18$, $w = 2$. Blocks: $[12, 2, 17, 0]$, $[1, 6, 13, 14]$, $[3, 0, 9, 15]$, $[10, 2, 7, 15]$, $[14, 17, 6, 15]$, $[7, 4, 8, 15]$, $[0, 15, 4, 13]$, $[13, 15, 1, 9]$, $[5, 14, 0, 13]$, $[13, 8, 16, 17]$, $[6, 7, 10, 0]$, $[16, 13, 7, 12]$, $[4, 2, 8, 9]$, $[6, 0, 16, 12]$, $[2, 6, 14, 16]$, $[6, 10, 8, 0]$, $[15, 2, 3, 5]$, $[13, 3, 6, 16]$, $[9, 11, 16, 10]$, $[13, 12, 15, 5]$, $[16, 2, 1, 8]$, $[4, 0, 1, 11]$, $[8, 3, 4, 6]$, $[12, 1, 3, 15]$, $[8, 7, 17, 9]$, $[4, 10, 17, 12]$, $[9, 14, 4, 12]$, $[0, 11, 7, 16]$, $[3, 10, 5, 13]$, $[1, 16, 9, 5]$, $[4, 5, 0, 2]$, $[2, 9, 10, 12]$, $[9, 12, 11, 13]$, $[14, 15, 17, 8]$, $[5, 11, 17, 1]$, $[5, 12, 6, 8]$, $[16, 5, 2, 14]$, $[5, 17, 3, 12]$, $[3, 9, 4, 16]$, $[5, 6, 11, 15]$, $[2, 13, 17, 4]$, $[9, 13, 10, 14]$, $[12, 14, 3, 11]$, $[16, 3, 7, 14]$, $[5, 7, 9, 6]$, $[13, 1, 7, 0]$, $[14, 1, 4, 11]$, $[8, 5, 1, 10]$, $[16, 15, 10, 11]$, $[2, 11, 14, 8]$, $[0, 12, 8, 11]$, $\{10, 14, 19, 4\}$, $\{2, a_1, 13, 7\}$, $\{15, a_1, 11, 4\}$, $\{a_0, 1, 8, 12\}$, $\{a_1, 10, 8, 13\}$, $\{7, 6, 2, 11\}$, $\{10, 14, 15, a_0\}$, $\{8, 15, 16, a_0\}$, $\{7, a_0, 12, 14\}$, $\{13, 8, a_1, 3\}$, $\{6, a_0, 1, 4\}$, $\{11, 10, a_0, 5\}$, $\{3, a_0, 0, 10\}$, $\{a_0, 6, 17, 9\}$, $\{0, 14, a_0, 3\}$, $\{8, 5, 14, a_1\}$, $\{2, 12, a_0, a_1\}$, $\{4, 17, 7, 3\}$, $\{16, 1, a_1, a_0\}$, $\{a_0, 0, 8, 9\}$, $\{0, 9, 14, a_1\}$, $\{14, 1, 17, 7\}$, $\{a_1, 15, 9, 6\}$, $\{11, 17, 9, a_1\}$, $\{1, 17, 10, 3\}$, $\{16, 10, 4, 12\}$, $\{17, 15, 16, a_0\}$, $\{1, 0, 15, 2\}$, $\{6, a_1, 11, 3\}$, $\{2, 1, 9, 3\}$, $\{a_0, 13, 11, 17\}$, $\{13, 4, a_0, 6\}$, $\{1, 6, 12, a_1\}$, $\{17, a_1, 0, 5\}$, $\{17, 16, a_1, 0\}$, $\{8, 11, 3, 7\}$, $\{7, a_0, 5, 9\}$, $\{2, 3, a_0, a_1\}$, $\{7, 10, a_1, 1\}$, $\{12, a_1, 15, 7\}$, $\{13, 11, 2, a_0\}$, $\{6, 10, 17, 2\}$, $\{16, a_1, 4, 5\}$, $\{4, a_0, 5, 7\}$.
10. $\lambda = 3$, $v = 19$, $w = 1$. Take an $S_2(2, 4, 19)$ which embeds a $P(19, 4, 1)$ and add the blocks $\{a_0, i, 4 + i, 10 + i\}$, $\{i, 18 + i, 11 + i, 16 + i\}$, $i \in \mathbb{Z}_{19}$.
11. $\lambda = 3$, $v = 21$, $w = 0$. Set $V = \mathbb{Z}_{17} \cup \{\infty_0, \infty_1, \infty_2, \infty_3\}$. Develop (mod 17) the base blocks $[\infty_0, 3, 1, 10]$, $[\infty_1, 5, 1, 8]$, $[\infty_2, 6, 1, 7]$, $[\infty_3, 4, 1, 2]$, $\{1, 3, 8, 12\}$, $\{1, 4, 5, 10\}$. Paste an $S_3(2, 4, 4)$ on $\{\infty_0, \infty_1, \infty_2, \infty_3\}$ which embeds a $P(4, 4, 1)$.
12. $\lambda = 3$, $v = 22$, $w = 2$. Take an $S_2(2, 4, 22)$ which embeds a $P(22, 4, 1)$ and add the blocks $\{12, 13, 14, 19\}$, $\{7, 10, 15, 19\}$, $\{8, 11, 16, 19\}$, $\{3, 9, 17, 19\}$, $\{0, 4, 18, 19\}$, $\{20, 2, 4, 11\}$, $\{12, 15, 16, 20\}$, $\{1, 10, 13, 20\}$, $\{4, 6, 17, 20\}$, $\{3, 5, 18, 20\}$, $\{0, 6, 16, 21\}$, $\{12, 17, 18, 21\}$, $\{2, 8, 13, 21\}$, $\{1, 7, 14, 21\}$, $\{5, 9, 15, 21\}$, $\{a_0, 1, 5, 19\}$, $\{a_0, 0, 8, 20\}$, $\{a_0, 3, 10, 21\}$, $\{a_0, 13, 15, 17\}$, $\{a_0, 14, 15, 18\}$, $\{a_0, 1, 9, 12\}$, $\{a_0, 4, 8, 12\}$, $\{a_0, 6, 11, 12\}$, $\{a_0, 0, 3, 13\}$, $\{a_0, 4, 7, 13\}$, $\{a_0, 0, 5, 14\}$, $\{a_0, 6, 10, 14\}$, $\{a_0, 1, 11, 17\}$, $\{a_0, 2, 5, 17\}$, $\{a_0, 8, 9, 18\}$, $\{a_0, 7, 11, 18\}$, $\{a_0, 2, 9, 15\}$, $\{a_0, 2, 7, 16\}$, $\{a_0, 3, 6, 16\}$, $\{a_0, 4, 10, 16\}$, $\{a_1, 2, 6, 19\}$, $\{a_1, 7, 9, 20\}$, $\{a_1, 4, 11, 21\}$, $\{a_1, 13, 16, 18\}$, $\{a_1, 14, 16, 17\}$, $\{a_1, 0, 2, 12\}$, $\{a_1, 3, 7, 12\}$, $\{a_1, 5, 10, 12\}$, $\{a_1, 5, 11, 13\}$, $\{a_1, 6, 9, 13\}$, $\{a_1, 3, 8, 14\}$, $\{a_1, 4, 9, 14\}$, $\{a_1, 0, 7, 17\}$, $\{a_1, 8, 10, 17\}$, $\{a_1, 1, 6, 18\}$, $\{a_1, 2, 10, 18\}$, $\{a_1, 0, 1, 16\}$, $\{a_1, 1, 8, 15\}$, $\{a_1, 4, 5, 15\}$, $\{a_1, 3, 11, 15\}$, $\{a_0, a_1, 19, 20\}$, $\{a_0, a_1, 19, 21\}$, $\{a_0, a_1, 20, 21\}$, $\{1, 2, 3, 4\}$, $\{0, 9, 10, 11\}$.
13. $\lambda = 3$, $v = 24$, $w = 0$. Base blocks: $[\infty, 1, 2, 4]$, $[1, 4, 8, 13]$, $[1, 7, 14, 22]$, $[1, 10, 20, 9]$, $\{1, 2, 7, 12\}$, $\{1, 3, 7, 10\}$.
14. $\lambda = 3$, $v = 27$, $w = 1$. Develop (mod 27) the following base blocks: $[7, 3, 1, 18]$, $[9, 1, 16, 5]$, $[6, 1, 15, 21]$, $[4, 1, 10, 17]$, $\{1, 6, 11, 14\}$, $\{a_0, 1, 10, 18\}$. To complete the $P(27, 4, 1)$ we must form the blocks covering the difference 1. Moreover, differences 1 and 2 must be covered twice more and difference 3 once more in the $S_3(2, 4, 28)$. Cover these

differences by $[1+3i, 2+3i, 3+3i, 4+3i]$, $\{2+3i, 3+3i, 4+3i, 5+3i\}$, $\{3+3i, 4+3i, 5+3i, 6+3i\}$, $i = 0, 1, \ldots, 8$ (arithmetic is $\pmod{27}$).

15. $\lambda = 3$, $v = 39$, $w = 1$. Put $V = \mathbb{Z}_{32} \cup \{\infty_i \mid i \in \mathbb{Z}_7\}$, $W = \{a_0\}$. Develop $\pmod{32}$ the base blocks: $[\infty_0, 0, 7, 6]$, $[\infty_1, 0, 3, 7]$, $[\infty_2, 0, 6, 11]$, $[\infty_3, 0, 2, 14]$, $[\infty_4, 0, 9, 19]$, $[\infty_5, 14, 0, 15]$, $[\infty_6, 11, 0, 13]$, $\{0, 4, 9, 10\}$, $\{0, 10, 12, 15\}$, $\{0, 4, 11, 23\}$, $\{a_0, 0, 3, 18\}$. Note that the differences 8 and 16 are missing. Using them construct $[i, 8+i, 16+i, 24+i]$, $[16+i, i, 24+i, 8+i]$, $\{i, 8+i, 16+i, 24+i\}$, $i = 0, 1, \ldots, 7$ (arithmetic is $\pmod{32}$). Finally, paste an $S_3(2, 4, 8)$ on $\{\infty_0, \infty_1, \ldots, \infty_6\} \cup \{a_0\}$ which embeds $P(7, 4, 1)$ on $\{\infty_0, \infty_1, \ldots, \infty_6\}$ (see step 3).
16. $\lambda = 3$, $v = 70$, $w = 2$. Let $X = \{x_1, x_2, \ldots, x_{18}\}$, $V = X \cup \mathbb{Z}_{52}$ and $W = \{a_0, a_1\}$. Construct an $S_3(2, 4, 20)$ $(X \cup W, \mathcal{D})$ which embeds a $P(18, 4, 1)$ on X (see step 9). First develop the following blocks $\pmod{52}$: $\{0, 8, 20, 31\}$, $\{0, 9, 24, 34\}$, $\{a_0, 0, 8, 20\}$, $\{a_1, 0, 23, 24\}$, $[x_1, 0, 5, 1]$, $[x_2, 0, 19, 2]$, $[x_3, 0, 21, 9]$, $[x_4, 0, 16, 10]$, $[x_5, 0, 9, 27]$, $[x_6, 0, 10, 21]$, $[x_7, 0, 23, 38]$, $[x_8, 0, 20, 28]$, $[x_9, 0, 25, 49]$, $[x_{10}, 0, 22, 15]$. Now add the blocks $[i, 13+i, 26+i, 39+i]$, $[13+i, 39+i, i, 26+i]$, $\{i, 13+i, 26+i, 39+i\}$ $i = 0, 1, \ldots, 12$.

    In order to complete the construction we need to form blocks of $\mathcal{B}$ covering each of the differences 1 to 7, 14, 16, 17, 19, 22 twice and the edges of $3K_{8,52}$ on $\{a_{11}, a_{12}, \ldots, a_{18}\} \cup \mathbb{Z}_{52}$. Moreover, some of these blocks must induce P_4s covering the differences 1, 2, 3, 14 and the edges of $K_{8,52}$ on $\{a_{11}, a_{12}, \ldots, a_{18}\} \cup \mathbb{Z}_{52}$.

    Construct the triples $\{22+i, i, 38+i\}$, $\{i, 17+i, 19+i\}$, $\{7+i, i, 3+i\}$, $\{i, 5+i, 6+i\}$, $i \in \mathbb{Z}_{52}$. It is easy to partition the set $\{\{22+i, i, 38+i\} \mid i \in \mathbb{Z}_{52}\}$ into two parts $\{\{\alpha_j, \beta_j, \gamma_j\} \mid j = 0, 1, \ldots, 12\}$ and $\{\{\alpha'_j, \beta'_j, \gamma'_j\} \mid j = 0, 1, \ldots, 12\}$ so that $F_1 = \{\{\alpha_j, \beta_j\} \mid j = 0, 1, \ldots, 12\}$ $F_2 = \{\{\alpha'_j, \beta'_j\} \mid j = 0, 1, \ldots, 12\}$ are two one-factors such that $F_1 \cup F_2$ covers $\{\{22+i, i\} \mid i \in \mathbb{Z}_{52}\}$ corresponding to the difference 22 on $\mathbb{Z}_{52}$. Now form blocks $[\alpha_j, x_{11}, \beta_j, \gamma_j]$, $[\alpha'_j, x_{12}, \beta'_j, \gamma'_j]$, $\{\alpha_j, \beta_j, \gamma_j, x_{12}\}$, $\{\alpha'_j, \beta'_j, \gamma'_j, x_{11}\}$, $j = 0, 1, \ldots, 12$.

    Repeat the same procedure with pairs $\{x_{13}, x_{14}\}$, $\{x_{15}, x_{16}\}$, $\{x_{17}, x_{18}\}$ and triples $\{i, 17+i, 19+i\}$, $\{7+i, i, 3+i\}$, $\{i, 5+i, 6+i\}$, $i \in \mathbb{Z}_{52}$, respectively.
17. $\lambda = 4$, $v = 6$, $w = 1$. Blocks: $[0, 4, 5, 1]$, $[4, 1, 0, 2]$, $[5, 2, 1, 3]$, $[5, 3, 2, 4]$, $[5, 0, 3, 4]$, $\{a_0, 4, 5, 2\}$, $\{a_0, 0, 1, 5\}$, $\{a_0, 0, 4, 3\}$, $\{a_0, 0, 5, 2\}$, $\{a_0, 0, 2, 3\}$, $\{a_0, 1, 4, 2\}$, $\{a_0, 1, 4, 3\}$, $\{a_0, 1, 5, 3\}$, $\{0, 1, 2, 3\}$.
18. $\lambda = 4$, $v = 7$, $w = 6$. (Note that this is not a minimum embedding, but we need this construction in the proof of Theorem 3.9.) Develop $\pmod 7$ the base block $[0, 4, 2, 1]$ and add the blocks $\{a_0, a_1, 1, 3\}$, $\{a_0, a_1, 4, 3\}$, $\{a_0, a_1, 2, 3\}$, $\{a_0, a_2, 5, 4\}$, $\{a_0, a_2, 6, 0\}$, $\{a_0, a_2, 4, 6\}$, $\{a_0, a_3, 2, 0\}$, $\{a_0, a_4, 0, 1\}$, $\{a_0, a_4, 2, 6\}$, $\{a_0, a_4, 3, 5\}$, $\{a_0, a_5, 1, 2\}$, $\{a_0, a_5, 0, 4\}$, $\{a_0, a_5, 5, 6\}$, $\{a_0, a_3, 1, 5\}$, $\{a_1, a_2, 5, 2\}$, $\{a_1, a_2, 0, 5\}$, $\{a_1, a_2, 1, 6\}$, $\{a_1, a_4, 0, 6\}$, $\{a_1, a_4, 1, 4\}$, $\{a_1, a_3, 2, 4\}$, $\{a_1, a_3, 0, 5\}$, $\{a_1, a_3, 3, 6\}$, $\{a_1, a_5, 1, 6\}$, $\{a_1, a_5, 2, 5\}$, $\{a_1, a_5, 0, 4\}$, $\{a_2, a_3, 2, 6\}$, $\{a_2, a_3, 0, 3\}$, $\{a_2, a_3, 1, 4\}$, $\{a_2, a_4, 0, 3\}$, $\{a_2, a_4, 1, 5\}$, $\{a_2, a_4, 2, 4\}$, $\{a_3, a_4, 1, 2\}$, $\{a_3, a_4, 3, 4\}$, $\{a_3, a_4, 5, 6\}$, $\{a_3, a_5, 0, 1\}$, $\{a_3, a_5, 3, 5\}$, $\{a_3, a_5, 4, 6\}$, $\{a_4, a_5, 0, 2\}$, $\{a_4, a_5, 3, 6\}$, $\{a_4, a_5, 4, 5\}$, $\{a_2, a_5, 1, 3\}$, $\{a_2, a_5, 2, 3\}$, $\{a_0, a_1, a_3, a_4\}$, $\{a_0, a_2, a_3, a_5\}$, $\{a_1, a_2, a_4, a_5\}$.
19. $\lambda = 4$, $v = 9$, $w = 1$. Take the blocks $\{a_0, x, y, t\}$ for each triple $\{x, y, t\}$ of a Steiner triple system of order 9 and paste an $S_3(2, 4, 9)$ which embeds a $P(9, 4, 1)$.
20. $\lambda = 4$, $v = 9$, $w = 4$ (note that this is not a minimum embedding, but we need this construction in the proof of Theorem 3.10). Blocks:
    - $[i, 4+i, 2+i, 5+i]$ for $i \in \mathbb{Z}_9$, $[0, 1, 2, 3]$, $[3, 4, 5, 6]$, $[6, 7, 8, 0]$;
    - $\{a_i, x, y, t\}$ for each triple $\{x, y, t\} \in \mathcal{R}_i$, where $\mathcal{R}_i$, $i = 0, 1, 2, 3$, are the four parallel classes of a Kirkman triple system on $\mathbb{Z}_9$;
    - $\{a_0, a_1, 0, 6\}$, $\{a_0, a_1, 1, 8\}$, $\{a_0, a_1, 3, 7\}$, $\{a_0, a_1, 4, 5\}$, $\{a_0, a_2, 0, 4\}$, $\{a_0, a_2, 1, 7\}$, $\{a_0, a_2, 2, 6\}$, $\{a_0, a_2, 3, 8\}$, $\{a_0, a_3, 0, 5\}$, $\{a_0, a_3, 2, 8\}$, $\{a_0, a_3, 3, 4\}$, $\{a_0, a_3, 6, 7\}$, $\{a_1, a_2, 0, 1\}$, $\{a_1, a_2, 2, 5\}$, $\{a_1, a_2, 3, 6\}$, $\{a_1, a_2, 4, 7\}$, $\{a_1, a_3, 0, 3\}$, $\{a_1, a_3, 1, 6\}$, $\{a_1, a_3, 2, 7\}$, $\{a_1, a_3, 5, 8\}$, $\{a_2, a_3, 0, 8\}$, $\{a_2, a_3, 1, 4\}$, $\{a_2, a_3, 2, 3\}$, $\{a_2, a_3, 5, 6\}$, $\{a_0, 2, 5, 1\}$, $\{a_1, 2, 8, 4\}$, $\{a_2, 5, 8, 7\}$, $\{a_3, 1, 4, 7\}$.
21. $\lambda = 4$, $v = 12$, $w = 1$. Blocks: $[8, 3, 7, 6]$, $[5, 6, 0, 1]$, $[4, 7, 5, 1]$, $[4, 6, 1, 8]$, $[4, 5, 2, 1]$, $[5, 8, 6, 11]$, $[8, 0, 3, 11]$, $[1, 4, 2, 6]$, $[3, 2, 10, 6]$, $[3, 9, 2, 7]$, $[9, 11, 0, 7]$, $[5, 10, 11, 4]$, $[6, 9, 5, 0]$, $[6, 3, 5, 11]$, $[4, 8, 7, 9]$, $[7, 10, 9, 1]$, $[7, 1, 10, 0]$, $[11, 1, 3, 10]$, $[4, 10, 8, 9]$, $[3, 4, 9, 0]$, $[4, 0, 2, 11]$, $[7, 11, 8, 2]$, $\{6, 11, 9, 10\}$, $\{7, 0, 2, 6\}$, $\{a_0, 9, 6, 1\}$, $\{0, 7, 8, 5\}$, $\{3, 8, 1, 2\}$, $\{a_0, 11, 7, 5\}$, $\{a_0, 7, 10, 3\}$, $\{a_0, 4, 3, 5\}$, $\{a_0, 1, 11, 8\}$, $\{a_0, 6, 2, 7\}$, $\{5, 8, 9, 3\}$, $\{a_0, 1, 11, 2\}$, $\{2, 4, 0, 3\}$, $\{a_0, 2, 9, 8\}$, $\{4, 6, 7, 10\}$, $\{a_0, 8, 6, 10\}$, $\{1, 5, 3, 7\}$, $\{10, 11, 0, 2\}$, $\{a_0, 8, 0, 4\}$, $\{a_0, 4, 10, 3\}$, $\{9, 1, 11, 3\}$, $\{a_0, 1, 9, 0\}$, $\{a_0, 4, 11, 7\}$, $\{11, 9, 6, 4\}$, $\{8, 0, 10, 1\}$, $\{a_0, 5, 2, 9\}$, $\{2, 10, 9, 5\}$, $\{8, 2, 10, 5\}$, $\{a_0, 10, 0, 5\}$, $\{a_0, 3, 0, 6\}$.
22. $\lambda = 4$, $v = 15$, $w = 1$. Blocks: $[4, 5, 12, 13]$, $[13, 2, 5, 14]$, $[4, 8, 7, 9]$, $[2, 6, 1, 8]$, $[13, 10, 9, 14]$, $[0, 3, 2, 12]$, $[14, 3, 13, 1]$, $[10, 7, 3, 1]$, $[11, 9, 4, 3]$, $[7, 0, 4, 11]$, $[10, 6, 11, 8]$, $[2, 9, 13, 5]$, $[5, 6, 9, 12]$, $[3, 5, 11, 13]$, $[12, 4, 13, 6]$, $[6, 12, 11, 14]$, $[2, 10, 5, 9]$, $[1, 14, 0, 9]$, $[12, 14, 13, 7]$, $[5, 8, 9, 3]$, $[1, 4, 6, 14]$, $[14, 2, 0, 6]$, $[1, 11, 3, 12]$, $[5, 1, 10, 8]$, $[8, 12, 0, 5]$, $[0, 13, 8, 2]$, $[14, 4, 10, 3]$, $[11, 10, 0, 1]$, $[7, 11, 0, 8]$, $[4, 7, 1, 12]$, $[10, 12, 7, 14]$, $[7, 6, 3, 8]$, $[11, 2, 1, 9]$, $[5, 7, 2, 4]$, $[6, 8, 14, 10]$, $\{3, 9, 7, 14\}$, $\{8, 12, 2, 3\}$, $\{7, 13, 10, 0\}$, $\{a_0, 6, 5, 7\}$, $\{a_0, 12, 11, 2\}$, $\{1, 2, 3, 4\}$, $\{1, 0, 2, 4\}$, $\{a_0, 0, 4, 13\}$, $\{3, 13, 6, 0\}$, $\{a_0, 10, 12, 0\}$, $\{14, 3, 11, 5\}$, $\{3, 0, 5, 7\}$, $\{a_0, 5, 10, 7\}$, $\{12, 6, 10, 9\}$, $\{a_0, 14, 11, 4\}$, $\{a_0, 14, 8, 11\}$, $\{10, 6, 2, 4\}$, $\{a_0, 14, 1, 5\}$, $\{5, 1, 11, 12\}$, $\{a_0, 2, 14, 8\}$, $\{a_0, 4, 8, 13\}$, $\{a_0, 6, 9, 3\}$, $\{0, 5, 14, 8\}$, $\{1, 8, 13, 9\}$, $\{a_0, 11, 9, 0\}$, $\{6, 7, 11, 2\}$, $\{a_0, 2, 7, 9\}$, $\{a_0, 3, 10, 2\}$, $\{14, 4, 9, 0\}$, $\{3, 8, 4, 10\}$, $\{13, 7, 6, 1\}$, $\{a_0, 12, 8, 1\}$, $\{12, 9, 10, 0\}$, $\{13, 11, 8, 7\}$, $\{a_0, 12, 13, 3\}$, $\{8, 4, 12, 9\}$, $\{a_0, 9, 7, 1\}$, $\{10, 11, 2, 13\}$, $\{12, 7, 14, 2\}$, $\{a_0, 6, 3, 0\}$, $\{9, 6, 11, 13\}$, $\{4, 10, 11, 5\}$, $\{a_0, 10, 13, 1\}$, $\{a_0, 4, 5, 6\}$, $\{5, 6, 0, 1\}$.
23. $\lambda = 4$, $v = 18$, $w = 1$. Blocks: $[8, 3, 9, 13]$, $[7, 6, 17, 11]$, $[7, 15, 8, 12]$, $[9, 14, 11, 7]$, $[2, 7, 12, 15]$, $[15, 14, 16, 2]$, $[4, 5, 1, 14]$, $[9, 2, 11, 3]$, $[0, 4, 9, 11]$, $[6, 4, 2, 10]$, $[7, 10, 9, 0]$, $[8, 1, 10, 15]$, $[7, 0, 1, 9]$, $[2, 17, 12, 11]$, $[17, 0, 15, 13]$,

[13, 3, 16, 9], [15, 6, 0, 10], [13, 2, 14, 7], [7, 8, 6, 1], [6, 11, 10, 13], [11, 13, 12, 6], [11, 1, 13, 0], [16, 1, 12, 14], [12, 10, 17, 14], [0, 8, 9, 15], [7, 4, 13, 14], [4, 15, 2, 3], [9, 17, 4, 1], [10, 16, 11, 0], [14, 3, 17, 5], [4, 8, 5, 10], [11, 15, 5, 7], [16, 7, 17, 8], [3, 12, 4, 10], [9, 7, 1, 2], [8, 11, 5, 13], [15, 3, 4, 14], [8, 2, 0, 12], [6, 13, 8, 14], [5, 9, 6, 10], [9, 12, 16, 6], [1, 17, 16, 15], [8, 16, 4, 11], [13, 16, 0, 14], [2, 12, 5, 0], [1, 3, 6, 2], [8, 10, 3, 5], [2, 5, 6, 14], [16, 5, 14, 10], [0, 3, 7, 13], [13, 17, 15, 1], $\{a_0, 3, 7, 10\}$, {13, 5, 15, 9}, {11, 3, 17, 14}, {0, 5, 6, 17}, {2, 17, 10, 7}, {10, 15, 4, 9}, {8, 3, 6, 0}, $\{a_0, 4, 16, 2\}$, $\{a_0, 11, 17, 4\}$, $\{a_0, 6, 8, 12\}$, {17, 9, 16, 5}, {5, 2, 13, 1}, $\{a_0, 9, 15, 6\}$, $\{a_0, 13, 10, 2\}$, {0, 1, 15, 3}, {3, 5, 7, 11}, $\{a_0, 7, 12, 5\}$, $\{a_0, 10, 7, 1\}$, $\{a_0, 13, 3, 17\}$, {4, 15, 6, 11}, {0, 5, 6, 4}, {13, 12, 4, 7}, $\{a_0, 4, 5, 7\}$, {4, 12, 14, 0}, {10, 14, 1, 11}, {10, 0, 14, 17}, {17, 8, 2, 10}, {8, 13, 16, 4}, {12, 6, 3, 17}, $\{a_0, 13, 15, 17\}$, $\{a_0, 2, 0, 1\}$, $\{a_0, 12, 11, 15\}$, $\{a_0, 0, 12, 3\}$, $\{a_0, 5, 0, 16\}$, {12, 13, 9, 10}, {2, 3, 9, 16}, {9, 12, 8, 14}, {7, 14, 8, 15}, $\{a_0, 11, 14, 0\}$, $\{a_0, 15, 6, 14\}$, {9, 12, 17, 5}, {1, 11, 12, 16}, $\{a_0, 17, 4, 9\}$, {11, 5, 2, 15}, {7, 16, 6, 3}, $\{a_0, 13, 10, 16\}$, {15, 10, 12, 16}, {17, 7, 16, 6}, {5, 15, 16, 3}, $\{a_0, 1, 5, 8\}$, {1, 3, 4, 12}, $\{a_0, 14, 2, 9\}$, {13, 2, 6, 4}, {5, 12, 13, 1}, {11, 16, 8, 2}, {16, 0, 4, 7}, {17, 2, 8, 0}, {1, 8, 4, 17}, {1, 11, 10, 3}, $\{a_0, 14, 3, 8\}$, $\{a_0, 11, 9, 8\}$, $\{a_0, 16, 1, 6\}$, {6, 1, 9, 14}.

24. $\lambda = 4, v = 21, w = 1$. Develop (mod 21) the blocks $\{a_0, 2, 6, 7\}$ and $\{0, 2, 8, 11\}$. Add the blocks $\{a_0, 1 + 6i, 8 + 6i, 15 + 6i\}, i = 0, 1, \ldots, 6$. Paste an $S_3(2, 4, 21)$ on $\mathbb{Z}_{21}$ which embeds a $P(21, 4, 1)$ (see step 11).
25. $\lambda = 4, v = 24, w = 1$. Take a 4-GDD of type $4^1 1^{24}$ on vertex set $\mathbb{Z}_v \cup \{\infty_i \mid i = 0, 1, 2, 3\}$. Let $G = \{\infty_i \mid i = 0, 1, 2, 3\}$ be the group of size 4. Replace each ∞_i with a_0 and take the blocks so obtained. Paste an $S_3(2, 4, 24)$ which embeds a $P(24, 4, 1)$ on vertex set $\mathbb{Z}_{24}$.
26. $\lambda = 4, v = 27, w = 1$. Paste an $S(2, 4, 28)$ to an $S_3(2, 4, 28)$ which embeds a $P(27, 4, 1)$ (see step 14).
27. $\lambda = 5, v = 6, w = 7$. Blocks: [0, 3, 1, 2], [4, 5, 0, 1], [5, 3, 4, 0], [3, 2, 4, 1], [0, 2, 5, 1], $\{a_0, , a_6, 21, 6\}$, $\{a_1, 2, a_2, 5\}$, $\{a_4, a_5, 1, 4\}$, $\{2, a_3, 5, a_0\}$, $\{1, a_5, 2, a_6\}$, $\{a_1, a_2, 4, a_3\}$, $\{a_1, a_3, 1, a_4\}$, $\{4, 2, a_3, 5\}$, $\{a_3, 3, a_5, a_0\}$, $\{a_4, a_6, 1, a_2\}$, $\{4, 5, a_6, 1\}$, $\{2, 3, a_1, a_6\}$, $\{3, a_4, a_6, 5\}$, $\{0, a_5, 4, a_0\}$, $\{3, a_0, a_2, 1\}$, $\{a_2, a_6, 1, 0\}$, $\{a_0, 0, 2, a_1\}$, $\{a_5, 3, 4, 0\}$, $\{a_0, 3, a_3, 1\}$, $\{1, a_1, 3, a_5\}$, $\{a_1, a_5, a_6, 4\}$, $\{2, 4, a_5, a_2\}$, $\{a_2, 3, 5, a_5\}$, $\{a_0, a_3, 1, a_5\}$, $\{a_4, a_5, a_6, 0\}$, $\{0, a_2, a_5, a_3\}$, $\{5, a_0, 1, a_1\}$, $\{3, a_0, a_4, a_2\}$, $\{2, a_2, 4, 3\}$, $\{a_0, a_1, a_4, 0\}$, $\{a_6, 5, a_0, a_5\}$, $\{a_2, 0, 4, a_0\}$, $\{3, a_3, a_6, 0\}$, $\{a_2, 0, 1, a_1\}$, $\{a_4, a_3, a_5, 2\}$, $\{a_0, 5, a_2, a_4\}$, $\{a_4, a_0, 4, 2\}$, $\{a_3, a_4, 4, a_1\}$, $\{a_1, a_5, a_3, 3\}$, $\{3, 4, a_4, a_6\}$, $\{a_3, a_1, 1, 4\}$, $\{2, 5, a_5, a_4\}$, $\{3, a_1, a_0, 5\}$, $\{5, a_3, 1, a_4\}$, $\{a_1, 0, a_4, 2\}$, $\{a_4, a_1, 3, a_6\}$, $\{2, a_5, a_0, a_6\}$, $\{a_3, 0, a_0, a_6\}$, $\{0, a_2, a_3, a_6\}$, $\{2, a_3, 0, a_4\}$, $\{5, 0, 3, a_4\}$, $\{a_6, a_3, 4, 5\}$, $\{a_4, 4, a_2, a_0\}$, $\{a_6, a_0, a_1, 4\}$, $\{5, 0, a_5, a_1\}$, $\{a_2, a_1, a_6, 2\}$, $\{a_3, 5, a_6, a_2\}$, $\{2, 3, a_2, a_3\}$, $\{5, a_1, a_5, a_2\}$, $\{1, a_2, a_4, a_5\}$.
28. $\lambda = 5, v = 7, w = 6$. Blocks: [4, 0, 6, 2], [0, 3, 1, 5], [5, 2, 3, 4], [3, 5, 6, 1], [5, 0, 1, 2], [3, 6, 4, 1], [0, 2, 4, 5], $\{a_1, 5, a_3, 2\}$, $\{a_5, 4, a_4, a_1\}$, $\{a_1, 5, a_5, 1\}$, $\{4, a_0, 5, a_3\}$, $\{2, 3, a_0, 1\}$, $\{6, a_2, a_3, 5\}$, $\{6, a_4, 0, 2\}$, $\{a_3, 6, a_0, a_5\}$, $\{0, a_0, a_3, a_5\}$, $\{6, a_0, 3, a_2\}$, $\{a_4, 3, 5, a_1\}$, $\{2, a_5, 4, a_2\}$, $\{2, 3, a_1, a_3\}$, $\{a_3, 1, 0, 6\}$, $\{4, a_4, a_0, 1\}$, $\{3, a_4, a_1, 6\}$, $\{a_1, a_5, 3, 2\}$, $\{2, a_0, a_1, 0\}$, $\{4, a_2, 1, a_1\}$, $\{5, 6, a_1, a_2\}$, $\{a_1, 4, a_5, 0\}$, $\{3, 0, 4, a_3\}$, $\{a_2, 5, 0, a_0\}$, $\{3, a_4, 4, a_2\}$, $\{a_1, 5, a_0, a_4\}$, $\{2, 5, 6, 4\}$, $\{a_5, 4, a_2, 1\}$, $\{a_5, 1, 2, a_4\}$, $\{6, a_4, a_5, a_0\}$, $\{a_6, a_3, 3, 4\}$, $\{a_4, 0, 6, 1\}$, $\{a_5, a_1, 6, 0\}$, $\{a_2, 0, a_4, 3\}$, $\{a_3, a_0, 4, 6\}$, $\{a_3, a_5, 5, 0\}$, $\{a_4, a_2, a_3, 5\}$, $\{a_1, 1, a_3, a_2\}$, $\{6, a_1, a_3, 1\}$, $\{a_5, a_2, a_3, 3\}$, $\{a_5, 6, a_2, 2\}$, $\{a_0, a_4, 5, a_5\}$, $\{a_3, a_4, 1, 4\}$, $\{a_2, 5, a_5, 1\}$, $\{a_3, a_0, 2, 3\}$, $\{1, a_0, a_2, 2\}$, $\{0, a_4, a_5, 3\}$, $\{a_0, a_4, 4, 5\}$, $\{a_0, 4, 6, a_1\}$, $\{a_1, a_2, 0, 4\}$, $\{a_4, 6, a_2, 2\}$, $\{a_1, 1, a_0, 3\}$, $\{2, a_3, a_1, a_4\}$, $\{a_0, 3, a_2, 0\}$, $\{0, a_0, a_2, a_1\}$, $\{a_0, 1, a_5, 2\}$, $\{5, a_5, 6, 3\}$, $\{a_4, a_2, 2, a_3\}$, $\{a_3, 0, a_4, 1\}$.
29. $\lambda = 5, v = 9, w = 4$. Paste an $S(2, 3, 13)$ to an $S_4(2, 3, 13)$ which embeds a $P(9, 4, 1)$ (see step 20).
30. $\lambda = 5, v = 10, w = 3$. Blocks: [1, 8, 3, 0], [0, 1, 2, 3], [6, 9, 7, 3], [6, 8, 9, 2], [2, 5, 8, 4], [0, 7, 4, 9], [8, 7, 2, 0], [3, 9, 5, 0], [5, 7, 1, 9], [8, 2, 6, 5], [5, 3, 1, 6], [7, 6, 0, 9], [3, 4, 0, 8], [2, 4, 5, 1], [3, 6, 4, 1], $\{7, a_2, 4, 5\}$, $\{a_0, 0, 3, 7\}$, $\{a_0, a_2, 3, 9\}$, $\{a_2, 0, 2, a_0\}$, $\{1, 5, a_1, 6\}$, $\{8, 7, a_0, 1\}$, $\{6, 4, a_0, 3\}$, $\{7, a_2, 1, 4\}$, $\{9, 5, a_0, a_2\}$, $\{2, 7, 4, a_1\}$, $\{6, 7, a_2, 8\}$, $\{5, 2, 0, a_2\}$, $\{a_2, 4, 9, 3\}$, $\{8, a_1, 3, a_0\}$, $\{8, 4, 9, 5\}$, $\{7, 2, 3, 8\}$, $\{a_2, 4, 6, a_0\}$, $\{7, a_2, 5, a_1\}$, $\{a_1, a_2, 2, 3\}$, $\{4, a_1, 0, 6\}$, $\{5, 8, a_0, a_1\}$, $\{a_2, a_1, 6, 0\}$, $\{a_2, 4, 8, 0\}$, $\{a_1, 1, 7, 8\}$, $\{2, 6, 9, a_0\}$, $\{6, 0, 4, 5\}$, $\{3, 5, 7, 6\}$, $\{a_1, 2, 3, 4\}$, $\{8, 6, a_2, 1\}$, $\{a_1, 9, 2, 6\}$, $\{a_2, 7, 2, 3\}$, $\{1, a_2, 2, 6\}$, $\{a_1, 7, 9, 0\}$, $\{5, 9, 0, 1\}$, $\{2, a_0, 1, 0\}$, $\{3, 5, 8, a_1\}$, $\{a_1, 4, a_0, 1\}$, $\{0, 6, 8, a_0\}$, $\{a_1, 1, 9, 3\}$, $\{a_0, 6, 7, a_1\}$, $\{4, a_1, 2, 9\}$, $\{a_0, 9, 4, 8\}$, $\{7, 4, a_0, 1\}$, $\{a_0, a_2, 5, 3\}$, $\{a_1, a_2, 1, 0\}$, $\{5, 2, 7, a_0\}$, $\{5, a_0, 0, a_1\}$, $\{8, a_1, 9, a_2\}$, $\{9, 8, a_2, 1\}$, $\{9, 1, 2, a_0\}$.
31. $\lambda = 5, v = 18, w = 7$. Put $V = \mathbb{Z}_{v-1} \cup \{\infty\}$ and develop (mod 17) the base blocks $[\infty, 1, 2, 5]$, [1, 3, 8, 12], [1, 12, 4, 11], $\{a_0, 1, 6, 7\}$, $\{a_1, 1, 4, 9\}$, $\{a_2, 1, 3, 5\}$. Add $\{\infty, a_0, a_1, a_2\}$ repeated three times. The result is an $S_3(2, 4, 21)$ $(V \cup \{a_0, a_1, a_2\}, \mathcal{D})$ which embeds a $P(18, 4, 1)$ on V. Now take two copies of a Kirkman triple system of order 21 on $V \cup \{a_0, a_1, a_2\}$. Partition the 20 parallel classes into 4 parts, each containing 5 classes. Denote by $\mathcal{P}_i$, $i = 0, 1, 2, 3$, the set of blocks covered by classes in part i. Associate to each $\mathcal{P}_i$ a new point a_{i+3} and form the block set $\mathcal{E}_i = \{\{a_{i+3}, x, y, t\} \mid \{x, y, t\} \in \mathcal{P}_i\}$. Let $\mathcal{E}$ be the set containing the blocks of $\cup_{i=0}^{3} \mathcal{E}_i$ and $\{a_3, a_4, a_5, a_6\}$ five-times repeated. Then $(V \cup \{a_0, a_1, \ldots, a_6\}, \mathcal{D} \cup \mathcal{E})$ is the required $S_5(2, 4, 25)$.
32. $\lambda = 5, v = 21, w = 4$. Embed a $P(21, 4, 1)$ into an $S_3(2, 4, 21)$ on $\mathbb{Z}_{21}$ (see step 11). Paste an $S_5(2, 4, 4)$ on $\{a_0, a_1, a_2, a_3\}$. Take a resolvable $S_2(2, 3, 21)$ on $\mathbb{Z}_{21}$ having the resolution classes $\mathcal{R}_j, j = 0, 1, \ldots, 19$. For each $i = 0, 1, 2, 3$, place the blocks $\{a_i, x, y, t\}, \{x, y, t\} \in \cup_{j=0}^{4} \mathcal{R}_{5i+j}$.
33. $\lambda = 5, v = 22, w = 3$. Embed a $P(22, 4, 1)$ into an $S_2(2, 4, 22)$. Take a Kirkman triple system on $\mathbb{Z}_{21}$ and partition its block set into four classes $\mathcal{C}_i, i = 0, 1, 2, 3$, such that $\mathcal{C}_3$ contains the blocks of one parallel classes and $\mathcal{C}_i, i = 0, 1, 2$, the blocks of three parallel classes. Construct $\{21, x, y, t\}$ for every $\{x, y, t\} \in \mathcal{C}_3$ and $\{a_i, x, y, t\}$ for every $\{x, y, t\} \in \mathcal{C}_i$, $i = 0, 1, 2$. At last add the block $\{21, a_0, a_1, a_2\}$ repeated three times. The result is an $S_3(2, 4, 25)$ which embeds a $P(22, 4, 1)$. Finally, paste an $S_2(2, 4, 25)$.
34. $\lambda = 5, v = 24, w = 1$. Paste an $S(2, 4, 25)$ on $\mathbb{Z}_{24} \cup \{a_0\}$ to an $S_4(2, 4, 25)$ which embeds a $P(24, 4, 1)$.

35. $\lambda = 5, v = 30, w = 7$. Take a 4-GDD of type 6^5. On each block place an $S_2(2, 4, 4)$ which embeds a $P(4, 4, 1)$. For each group, embed a $P(6, 4, 1)$ into an $S_4(2, 4, 7)$ having a_6 as the new vertex. Now, take a resolvable 3-GDD of type 6^5 having the same groups as the previous 4-GDD and $\mathcal{R}_i$, $i = 0, 1, \ldots, 11$, as parallel classes. For $i = 0, 1, \ldots, 5$, construct the following blocks (each repeated two times): $\{a_i, x, y, t\}$, $\{x, y, t\} \in \mathcal{R}_{2i} \cup \mathcal{R}_{2i+1}$. Paste an $S_4(2, 4, 7)$ on $\{a_0, a_1, \ldots, a_6\}$. The result is an $S_4(2, 4, 37)$ which embeds a $P(30, 4, 1)$. Finally, paste an $S(2, 4, 37)$.
36. $\lambda = 5, v = 31, w = 6$. Let $V = \mathbb{Z}_{30} \cup \{\infty\}$ and $W = \{a_0, a_1, \ldots, a_5\}$. Take a 4-GDD of type 6^5 on $\mathbb{Z}_{30}$ having groups G_i, $i = 1, 2, \ldots, 5$. On each block place an $S_2(2, 4, 4)$ which embeds a $P(4, 4, 1)$. For $i = 1, 2, \ldots, 5$, place an $S_4(2, 4, 7)$ on $G_i \cup \{\infty\}$ which embeds a $P(7, 4, 1)$. Paste an $S_4(2, 4, 7)$ on $W \cup \{\infty\}$.

    Take a resolvable 3-GDD of type 6^5 having groups $G_1, G_2, \ldots, G_5$ and parallel classes $\mathcal{R}_j$, $j = 0, 1, \ldots, 11$. For $i = 0, 1, \ldots, 5$ take the blocks $\{a_i, x, y, t\}$, for every $\{x, y, t\} \in \cup_{j=0}^{1} \mathcal{R}_{2i+j}$, repeated twice. The result is an $S_4(2, 4, 37)$ which embeds a $P(31, 4, 1)$. Paste an $S(2, 4, 37)$.
37. $\lambda = 5, v = 42, w = 7$. Take a 4-GDD of type 6^7 on $\mathbb{Z}_{42}$ having groups $G_1, G_2, \ldots, G_7$. On each block place an $S_2(2, 4, 4)$ which embeds a $P(4, 4, 1)$. For $i = 1, 2, \ldots, 7$, place an $S_4(2, 4, 7)$ on $G_i \cup \{a_6\}$ which embeds a $P(6, 4, 1)$ on G_i. Now, take a resolvable 3-GDD of type 6^7 having groups $G_1, G_2, \ldots, G_7$ and parallel classes $\mathcal{R}_i$, $i = 0, 1, \ldots, 17$. For $i = 0, 1, \ldots, 5$, construct the blocks: $\{a_i, x, y, t\}$, $\{x, y, t\} \in \mathcal{R}_{3i} \cup \mathcal{R}_{3i+1} \cup \mathcal{R}_{3i+2}$. Paste an $S_4(2, 4, 7)$ on $\{a_0, a_1, \ldots, a_6\}$ and the blocks of a 4-GDD of type 6^8 with groups $\{a_0, a_1, \ldots, a_6\}, G_1, G_2, \ldots, G_7$. The result is an $S_4(2, 4, 49)$ which embeds a $P(42, 4, 1)$. Paste an $S(2, 4, 49)$.
38. $\lambda = 5, v = 43, w = 6$. Take a 4-GDD of type 6^7 on $\mathbb{Z}_{42}$ having groups $G_1, G_2, \ldots, G_7$. On each block place an $S_2(2, 4, 4)$ which embeds a $P(4, 4, 1)$. For $i = 1, 2, \ldots, 7$, place an $S_4(2, 4, 7)$ on $G_i \cup \{\infty\}$ which embeds a $P(7, 4, 1)$. Paste an $S_4(2, 4, 7)$ on $\{\infty\} \cup \{a_0, a_1, \ldots, a_5\}$.

    Take a resolvable 3-GDD of type 6^7 having groups $G_1, G_2, \ldots, G_7$ and parallel classes $\mathcal{R}_j$, $j = 0, 1, \ldots, 17$. For $i = 0, 1, \ldots, 5$, construct the blocks $\{a_i, x, y, t\}$, $\{x, y, t\} \in \cup_{j=0}^{2} \mathcal{R}_{3i+j}$. Now, take the blocks of a 4-GDD of type 6^8 having groups $\{a_0, a_1, \ldots, a_5\}, G_1, G_2, \ldots, G_7$. The result is an $S_4(2, 4, 49)$ which embeds a $P(43, 4, 1)$. Paste an $S(2, 4, 49)$.
39. $\lambda = 5, v = 54, w = 7$. Take a 4-GDD of type 6^9 on $\mathbb{Z}_{54}$ having groups $G_1, G_2, \ldots, G_9$. On each block place an $S_3(2, 4, 4)$ which embeds a $P(4, 4, 1)$. For $i = 1, 2, \ldots, 9$, place an $S_4(2, 4, 7)$ on $G_i \cup \{a_6\}$ which embeds a $P(6, 4, 1)$ on G_i. Now, take a resolvable 3-GDD of type 6^9 having groups $G_1, G_2, \ldots, G_9$ and parallel classes $\mathcal{R}_i$, $i = 0, 1, \ldots, 23$. For $i = 0, 1, \ldots, 5$, construct the blocks: $\{a_i, x, y, t\}$, $\{x, y, t\} \in \mathcal{R}_{4i} \cup \mathcal{R}_{4i+1} \cup \mathcal{R}_{4i+2} \cup \mathcal{R}_{4i+3}$. Paste an $S_4(2, 4, 7)$ on $\{a_0, a_1, \ldots, a_6\}$. The result is an $S_4(2, 4, 61)$ which embeds a $P(54, 4, 1)$. Paste an $S(2, 4, 61)$.
40. $\lambda = 5, v = 66, w = 7$. Take a 4-GDD of type $6^1 12^5$ on $\mathbb{Z}_{66}$. Let $G = \mathbb{Z}_6$ be the group of size 6 and let $G_1, G_2, \ldots, G_5$ be the groups of size 12. On each block place an $S_3(2, 4, 4)$ which embeds a $P(4, 4, 1)$. Embed a $P(6, 4, 1)$ on G into an $S_5(2, 4, 13)$ on $G \cup \{a_i \mid i = 0, 1, \ldots, 6\}$ (see step 27). For $i = 1, 2, \ldots, 5$, place an $S_4(2, 4, 13)$ on $G_i \cup \{a_6\}$ which embeds a $P(12, 4, 1)$ on G_i (see step 21).

    Now, take a resolvable 3-GDD of type 12^5 having groups $G_1, G_2, \ldots, G_5$ and parallel classes $\mathcal{R}_i$, $i = 0, 1, \ldots, 23$. For $i = 0, 1, \ldots, 5$, construct the blocks $\{a_i, x, y, t\}$ for every $\{x, y, t\} \in \cup_{j=0}^{3} \mathcal{R}_{j+4i}$.

    Take a 4-GDD of type $19^1 1^{60}$ having $H = \{a_0, a_1, \ldots, a_6\} \cup \{\infty_{ij} \mid (i, j) \in \mathbb{Z}_6 \times \mathbb{Z}_2\}$ as the group of size 19 and the elements of $\cup_{i=1}^{5} G_i$ as groups of size 1. For each $(i, j) \in \mathbb{Z}_6 \times \mathbb{Z}_2$, replace ∞_{ij} with i and take the blocks so obtained.
41. $\lambda = 5, v = 78, w = 7$. Take a 4-GDD of type $6^1 12^6$ on $\mathbb{Z}_{78}$. Let $G = \mathbb{Z}_6$ be the group of size 6 and let $G_1, G_2, \ldots, G_6$ be the groups of size 12. On each block place an $S_3(2, 4, 4)$ which embeds a $P(4, 4, 1)$. Embed a $P(6, 4, 1)$ on G into an $S_5(2, 4, 13)$ on $G \cup \{a_i \mid i = 0, 1, \ldots, 6\}$ (see step 27). For $i = 1, 2, \ldots, 6$, place an $S_4(2, 4, 13)$ on $G_i \cup \{a_6\}$ which embeds a $P(12, 4, 1)$ on G_i (see step 21).

    Now, take a resolvable 3-GDD of type 12^6 having groups $G_i, G_2, \ldots, G_6$ and parallel classes $\mathcal{R}_i$, $i = 0, 1, \ldots, 29$. For each $j \in G = \mathbb{Z}_6$, construct the blocks $\{j, x, y, t\}$, $\{x, y, t\} \in \mathcal{R}_j$. Using the triples in the remaining parallel classes construct $\{a_i, x, y, t\}$, $\{x, y, t\} \in \mathcal{R}_{4i+6} \cup \mathcal{R}_{4i+7} \cup \mathcal{R}_{4i+8} \cup \mathcal{R}_{4i+9}$, $i = 0, 1, \ldots, 5$. Finally, paste the blocks of a 4-GDD of type $13^1 1^{72}$ having $G \cup \{a_0, a_1, \ldots, a_6\}$ as the group of size 13 and the elements of $\cup_{i=1}^{6} G_i$ as groups of size 1.
42. $\lambda = 6, v = 6, w = 0$. Blocks [3, 4, 5, 2], [3, 1, 2, 4], [5, 0, 2, 3], [3, 5, 1, 0], [3, 0, 4, 1], {4, 1, 5, 3}, {3, 0, 1, 2}, {1, 5, 2, 3}, {3, 2, 0, 4}, {5, 0, 1, 4}, {3, 4, 5, 0}, {2, 4, 0, 5}, {2, 1, 4, 5}, {0, 4, 1, 2}, {1, 0, 5, 2}.
43. $\lambda = 6, v = 15, w = 0$. Blocks: [1, 12, 11, 2], [10, 11, 13, 4], [9, 3, 12, 6], [13, 0, 1, 5], [5, 8, 7, 3], [13, 14, 0, 2], [2, 3, 14, 4], [1, 2, 13, 3], [3, 5, 7, 4], [10, 0, 8, 3], [2, 4, 9, 0], [6, 4, 10, 12], [13, 10, 7, 14], [13, 7, 11, 5], [0, 3, 1, 9], [4, 8, 11, 3], [6, 7, 9, 10], [2, 8, 9, 11], [13, 8, 10, 1], [14, 12, 5, 10], [6, 0, 11, 14], [7, 2, 10, 3], [9, 14, 10, 6], [7, 0, 5, 13], [2, 9, 6, 14], [5, 2, 12, 7], [3, 4, 11, 1], [14, 8, 1, 4], [14, 1, 6, 11], [8, 12, 9, 13], [14, 2, 6, 8], [14, 5, 6, 3], [13, 12, 0, 4], [12, 4, 5, 9], [6, 13, 1, 7], {11, 10, 0, 1}, {12, 3, 1, 8}, {4, 10, 7, 2}, {5, 2, 1, 9}, {3, 5, 7, 11}, {8, 0, 7, 6}, {12, 4, 14, 7}, {4, 2, 0, 7}, {5, 10, 14, 0}, {1, 0, 2, 10}, {8, 3, 5, 9}, {11, 6, 9, 13}, {4, 8, 9, 1}, {10, 13, 8, 14}, {3, 11, 12, 14}, {9, 11, 2, 12}, {6, 8, 0, 11}, {11, 14, 4, 5}, {1, 12, 7, 14}, {2, 10, 0, 12}, {11, 6, 12, 10}, {2, 8, 11, 13}, {3, 10, 6, 2}, {14, 3, 11, 13}, {12, 2, 8, 0}, {4, 3, 0, 14}, {9, 10, 8, 5}, {3, 12, 14, 0}, {8, 13, 4, 6}, {6, 5, 0, 4}, {9, 13, 14, 0}, {13, 11, 6, 12}, {3, 0, 6, 12}, {8, 12, 0, 5}, {5, 9, 13, 2}, {1, 5, 14, 9}, {3, 12, 1, 13}, {6, 2, 1, 3}, {2, 3, 6, 5}, {12, 9, 13, 7}, {8, 5, 11, 0}, {2, 11, 5, 12}, {5, 13, 14, 2}, {14, 1, 7, 8}, {1, 9, 11, 4}, {13, 3, 7, 10}, {12, 8, 4, 10}, {0, 3, 9, 13}, {13, 5, 8, 6}, {12, 7, 14, 9}, {13, 2, 4, 6}, {1, 12, 10, 5}, {12, 1, 13, 4}, {7, 0, 11, 1}, {2, 4, 8, 14}, {2, 10, 7, 11}, {3, 10, 4, 13}, {10, 3, 8, 9}, {7, 0, 1, 6}, {9, 0, 4, 7}, {4, 5, 10, 11}, {6, 5, 1, 10}, {7, 3, 9, 11}, {1, 9, 10, 14}, {10, 9, 11, 0}, {6, 1, 5, 4}, {12, 6, 7, 8}, {9, 7, 4, 6}, {14, 7, 8, 11}, {7, 8, 1, 2}.

44. $\lambda = 6$, $v = 18$, $w = 0$. Blocks: [11, 1, 9, 10], [17, 1, 0, 4], [13, 3, 4, 1], [4, 15, 2, 7], [16, 13, 15, 10], [0, 17, 10, 14], [6, 7, 13, 8], [7, 0, 11, 17], [6, 0, 8, 4], [10, 1, 8, 11], [15, 17, 14, 4], [15, 3, 12, 4], [9, 12, 5, 15], [5, 10, 11, 4], [12, 8, 9, 15], [13, 4, 17, 9], [1, 13, 12, 15], [10, 6, 8, 3], [15, 1, 16, 5], [13, 9, 5, 8], [5, 14, 15, 0], [0, 14, 1, 3], [0, 2, 3, 7], [5, 1, 6, 12], [2, 13, 11, 12], [17, 12, 2, 1], [8, 14, 12, 1], [17, 7, 8, 2], [13, 6, 2, 16], [2, 9, 4, 5], [17, 6, 5, 3], [6, 16, 17, 8], [2, 17, 3, 6], [10, 16, 7, 9], [5, 17, 13, 0], [11, 15, 16, 3], [2, 5, 11, 16], [9, 14, 2, 10], [11, 7, 15, 6], [13, 10, 12, 0], [1, 7, 10, 3], [5, 7, 12, 16], [4, 6, 11, 9], [9, 6, 14, 11], [3, 9, 0, 5], [4, 7, 14, 16], [4, 2, 11, 3], [15, 8, 16, 9], [8, 10, 0, 3], [10, 4, 16, 0], [3, 14, 13, 5], {16, 11, 2, 6}, {11, 12, 16, 17}, {1, 16, 3, 7}, {5, 4, 13, 16}, {16, 9, 1, 11}, {0, 11, 15, 8}, {0, 11, 3, 9}, {4, 12, 17, 3}, {9, 3, 13, 15}, {15, 7, 11, 3}, {14, 7, 16, 17}, {13, 10, 8, 15}, {3, 7, 9, 17}, {6, 1, 9, 12}, {9, 2, 16, 1}, {3, 11, 13, 1}, {1, 8, 9, 17}, {11, 14, 4, 8}, {8, 16, 17, 3}, {15, 1, 7, 5}, {13, 14, 2, 9}, {10, 14, 4, 16}, {1, 16, 4, 8}, {3, 4, 5, 9}, {16, 14, 3, 6}, {0, 8, 10, 2}, {15, 6, 3, 10}, {12, 4, 7, 10}, {8, 9, 4, 7}, {11, 10, 7, 13}, {7, 16, 0, 2}, {9, 10, 13, 17}, {0, 12, 17, 2}, {13, 6, 14, 17}, {1, 5, 3, 8}, {3, 10, 16, 15}, {4, 6, 7, 12}, {3, 7, 13, 14}, {14, 9, 5, 10}, {13, 4, 5, 7}, {2, 11, 12, 15}, {13, 1, 11, 17}, {2, 10, 5, 7}, {3, 12, 14, 2}, {11, 12, 14, 6}, {15, 0, 6, 9}, {11, 5, 6, 7}, {5, 8, 2, 16}, {12, 0, 16, 9}, {14, 0, 9, 16}, {0, 14, 5, 6}, {7, 12, 14, 9}, {13, 4, 5, 0}, {6, 1, 5, 12}, {14, 10, 11, 5}, {14, 11, 4, 1}, {8, 17, 2, 9}, {6, 16, 0, 1}, {12, 2, 13, 10}, {10, 2, 5, 6}, {10, 11, 5, 17}, {10, 15, 9, 17}, {0, 8, 12, 13}, {3, 2, 5, 8}, {8, 11, 13, 0}, {13, 11, 8, 16}, {13, 2, 14, 1}, {17, 1, 6, 10}, {17, 1, 4, 10}, {2, 11, 4, 0}, {13, 9, 6, 7}, {7, 15, 2, 1}, {4, 8, 6, 15}, {8, 14, 2, 15}, {17, 14, 15, 16}, {11, 15, 5, 17}, {13, 7, 14, 15}, {8, 12, 5, 14}, {9, 7, 0, 1}, {17, 6, 13, 15}, {4, 6, 9, 2}, {11, 17, 7, 0}, {4, 10, 16, 12}, {12, 0, 3, 10}, {9, 12, 3, 11}, {5, 0, 15, 1}, {12, 11, 14, 8}, {6, 15, 10, 2}, {8, 7, 10, 12}, {2, 15, 0, 1}, {12, 5, 17, 16}, {3, 8, 4, 6}, {17, 14, 2, 3}, {6, 7, 0, 12}, {15, 4, 12, 17}, {4, 14, 0, 15}, {1, 4, 2, 13}, {3, 16, 12, 13}, {0, 13, 16, 6}, {10, 14, 1, 6}, {14, 1, 7, 8}, {17, 5, 7, 8}.

45. $\lambda = 6$, $v = 27$, $w = 0$. Set $V = \mathbb{Z}_{23} \cup \{\infty_0, \infty_1, \infty_2, \infty_3\}$. Develop (mod 23), $[\infty_0, 0, 7, 10]$, $[\infty_1, 1, 2, 4]$, $[\infty_2, 1, 9, 18]$, $[\infty_3, 1, 12, 2]$, [3, 9, 14, 18], {1, 12, 8, 3}, $\{\infty_1, 1, 2, 9\}$, $\{\infty_2, 1, 5, 11\}$, $\{\infty_3, 1, 4, 6\}$. Add the blocks of an $S_6(2, 4, 4)$ which embeds a $P(4, 4, 1)$ on the vertex set $\{\infty_0, \infty_1, \infty_2, \infty_3\}$ and the blocks of an $S_3(2, 4, 24)$ on the vertex set $\mathbb{Z}_{23} \cup \{\infty_0\}$.

46. $\lambda = 9$, $v = 6$, $w = 2$. Develop (mod 5) the base blocks $[\infty, 0, 1, 3]$, $\{a_0, \infty, 1, 3\}$, $\{a_1, \infty, 1, 2\}$, $\{a_0, a_1, 1, 2\}$, $\{a_0, 1, 2, 4\}$, $\{a_1, 1, 2, 3\}$. Add the block {1, 2, 3, 4} repeated twice and the following ones $\{a_0, a_1, \infty, i\}$ for $i = 1, 2, 3, 4$, $\{a_{,}0, 1, 4\}$, $\{a_1, 0, 1, 3\}$, $\{a_0, 0, 2, 3\}$, $\{a_1, 0, 2, 4\}$, $\{\infty, 0, 1, 2\}$, $\{\infty, 0, 3, 4\}$.

47. $\lambda = 9$, $v = 10$, $w = 2$. Embed a $P(10, 4, 1)$ into an $S_2(2, 4, 10)$ on $\mathbb{Z}_9 \cup \{\infty\}$. Develop (mod 9) the base blocks $\{a_0, a_1, 0, 1\}$, $\{a_0, \infty, 0, 4\}$, $\{a_1, \infty, 0, 1\}$, $\{a_0, 0, 3, 4\}$, $\{a_1, 0, 1, 2\}$, $\{\infty, 0, 2, 4\}$, {0, 2, 7, 6}, {0, 2, 7, 6}. Add the following blocks (each repeated twice): $\{a_0, 1, 4, 7\}$, $\{a_0, 2, 5, 8\}$, $\{a_0, 3, 6, 0\}$, $\{a_1, 1, 4, 7\}$, $\{a_1, 2, 5, 8\}$, $\{a_1, 3, 6, 0\}$.

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